Global Fluctuations of Random Matrices and The Second-Order Cauchy Transform

by

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Abstract

In this thesis we study the global fluctuations of random matrices (i.e., the covariance of two traces) from a second-order free probability perspective, putting a particular emphasis on block Gaussian matrices. Our main contributions are threefold.

First, we provide a formula for the second-order Cauchy transform of block Gaussian matrices in terms of the corresponding (matrix-valued) Cauchy transform. In order to do this, we introduce a (matrix-valued) second-order conditional expectation, in the spirit of the conditional expectations in operator-valued free probability theory.

Second, we establish a criterion, based on the positivity of a cluster function, that guarantees the existence of a particular integral representation for the second-order Cauchy transform. This approach is based on some relations between the moments and fluctuation moments of a measure on the plane.

Finally, we prove that under some conditions the covariance of resolvents converges to the second-order Cauchy transform. In fact, these conditions also ensure the convergence of the covariance of analytic linear statistics to a certain contour integral depending on the second-order Cauchy transform. In this context, we make explicit the relation between the notion of second-order limit distribution and the asymptotic Gaussianity of continuously differentiable linear statistics.

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Chapter 1

Introduction

In general terms, random matrix theory is the area of mathematics concerned with the study of matrix-valued random variables, a.k.a. random matrices. Of particular interest are their spectral properties such as the eigenvalue/eigenvector distribution, the eigenvalue spacing distribution, the largest eigenvalue distribution, the probability of singularity for discrete random matrices, etcetera. In the words of Mehta [29]: "Random matrices first appeared in mathematical statistics in the 1930s but did not attract much attention at the time." In fact, the works of Wishart, Fisher, and coauthors focused mainly on sample covariance matrices [62, 16]. In the 50's, the study of the asymptotic behavior of the empirical eigenvalue distribution of more general selfadjoint random matrices came into the picture. In a series of papers [59, 60, 61], Wigner showed that the empirical eigenvalue distribution of certain selfadjoint random matrices converges, when the dimension goes to infinity, to the semicircle distribution. Since then, random matrix theory has played major roles in many applied and theoretical areas of mathematics, going from the analysis of wireless communication systems [52], to the somehow enigmatic predictions about the statistical behavior of the zeroes (on the critical line) of the Riemann zeta function found by Montgomery [38].

In the 80's, Voiculescu started what today is known as free probability theory [57]. In a sentence, free probability theory is an analogue of probability theory where the algebra of random variables and the expectation are replaced by a non-commutative algebra and a unital linear functional, respectively. Without doubt, one of the main features of classical probability theory is the systematic use of the notion of independence. In free probability theory, the notion of freeness replaces that of independence and plays the same protagonist role. Conforming to its original purposes, this theory led to outstanding advances in the study of operator algebras, see, e.g., [58]. Furthermore, it proved to be very useful in the study of the asymptotic behavior of random matrices [55, 18, 1, 5]. One of the first achievements of free probability theory in connection to random matrix theory is Voiculescu's theorem: independent Gaussian matrices are asymptotically free [56, 53]. Since then, many results in the literature have been devoted to establishing the asymptotic freeness of different random matrix ensembles [54, 11, 32], or to exploiting the implications of this asymptotic freeness [5, 49].

An important aspect of free probability theory comes from its combinatorial facet, with the non-crossing partitions in its heart [40]. These combinatorial objects are also at the core of the so-called operator-valued free probability theory [57, 50]. In this extension of free probability the expectation is replaced by a conditional expectation that may take values in a non-commutative complex algebra. It is important to remark that conditional expectations taking values in spaces of complex matrices are particularly useful in random matrix theory [5, 51]. This is the case, at least partially, because any non-commutative polynomial in non-commutative random variables can be *linearized* into a matrix of non-commutative random variables [18, 1, 23]. When translated into random matrix theory terms, this fact emphasizes the relevance of block random matrices, which are already interesting from an applied perspective [42, 14, 15].

In operator-valued free probability theory, the so-called operator-valued semicircular variables play an outstanding role. On one hand, they are the counterpart of Gaussian random variables in many aspects; on the other, they encode the asymptotic behavior of selfadjoint block Gaussian matrices. From a combinatorial point of view, the expectation of a product of semicircular variables can be written in terms of non-crossing pairings. About a decade ago, Mingo and Nica [31] extended this combinatorial treatment to the study of the global fluctuations (i.e., the covariance of two traces) of selfadjoint Gaussian matrices. In particular, they demonstrated that these global fluctuations depend on another type of pairings, the non-crossing annular pairings. In order to systematize the combinatorial treatment of the global fluctuations of random matrices, Mingo and Speicher introduced a theory called second-order free probability [36, 34, 12]. Despite the developments in the last decade [37, 33, 43, 25], many statements in free probability and random matrix theory still lack of a second-order counterpart. It is the purpose of this thesis to study the global fluctuations of random matrices from a second-order free probability perspective, putting a particular emphasis on block Gaussian matrices.

Before discussing the specific problems approached in this thesis, we would like to remark that there are many key contributions to the study of the global fluctuations of not necessarily (block) Gaussian random matrices. For example: Diaconis and Shahshahani's work on the unitary, orthogonal, symplectic, and symmetric groups [13];

Johansson's study of Hermitian matrices with eigenvalue distributions determined by a potential [26]; Bai and Silverstein's work on sample covariance matrices [3], and further extensions [19, 20, 39]; Anderson and Zeitouni's contributions to the case of band matrix models [2]; and those summarized in Pastur and Shcherbina's book [41].

In this thesis we address the following three problems. Let $(X_N : N \in \mathbb{N})$ be a random matrix ensemble, i.e., X_N is an $N \times N$ random matrix for each $N \in \mathbb{N}$. The second-order moments of $(X_N : N \in \mathbb{N})$ equal, by definition,

$$\alpha_{m,n} = \lim_{N\to\infty} \operatorname{Cov}\left(\operatorname{Tr}\left(X_N^m\right), \operatorname{Tr}\left(X_N^n\right)\right),$$

for $m, n \in \mathbb{N}$. When all the second-order moments exist, the second-order Cauchy transform of $(X_N : N \in \mathbb{N})$ is defined as the generating function G_2 given by

$$G_2(z,w) = \sum_{m,n\geq 0} \frac{\alpha_{m,n}}{z^{m+1}w^{n+1}}.$$

In this context there are three natural problems to consider.

P1. Find an effective way to compute G_2.

P2. Determine whether or not G_2 can be represented as

$$G_2(z,w) = \int_{\mathbb{R}^2} \frac{1}{z-t_1}\frac{1}{z-t_2}\frac{1}{w-t_1}\frac{1}{w-t_2}\mathrm{d}\nu(t_1,t_2), \tag{1.1}$$

for some finite measure ν on $\mathbb{R}^2$.

P3. Find conditions on the random matrix ensemble that guarantee that

$$G_2(z,w) = \lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left((z\mathrm{I}_N - X_N)^{-1}\right), \mathrm{Tr}\left((w\mathrm{I}_N - X_N)^{-1}\right)\right),$$

where I_N denotes the $N \times N$ identity matrix.

Before continuing, let us remark on some of the implications of these problems. The second-order Cauchy transform G_2 carries all the second-order moments of the random matrix ensemble. Therefore, a technique to compute G_2 provides, in principle, a way to compute all the second-order moments. Moreover, such a technique might reveal the analytical properties of G_2, linking to our second problem as a second-order Cauchy transform satisfying (1.1) is necessarily analytic on $(\mathbb{C}\backslash\mathbb{R})^2$. Among other applications, the integral representation in (1.1) leads to a similar expression for the limit of the covariance of linear statistics. The third problem asks whether or not the second-order Cauchy transform and its underlying analytic theory describe the corresponding random matrix theory phenomena. The problems have come full circle.

Of course, solving these three problems for any random matrix ensemble is far from tractable at the moment. Thus, we should content ourselves with solving them for as many ensembles as possible. In order to get a taste of the difficulties ahead, let us discuss some results regarding the analogous problems in the first-order case.

In [5], Belinschi et al. provide an efficient numerical method to compute the Cauchy transform of any selfadjoint non-commutative polynomial evaluated in asymptotically free selfadjoint random matrices. In fact, this approach extends to the analysis of non-commutative rational functions as pointed out in [23]. Their method is heavily based on the linearization technique [18, 1] and recent developments in operator-valued free probability theory [6, 5]. It is important to remark that in the particular case of a selfadjoint non-commutative polynomial evaluated at independent selfadjoint Gaussian matrices, the main result in [22] provides a very efficient method to compute the desired Cauchy transform.

Let $(\beta_n : N \in \mathbb{N})$ be a sequence of real numbers. Recall that the Hamburger moment problem is solvable, i.e., $(\beta_n : N \in \mathbb{N})$ is a sequence of moments of a positive measure, if and only if

$$\sum_{i,j\geq 0} \beta_{i+j} c_i \overline{c_j} \geq 0$$

for every sequence $(c_i)_{i\geq 0}$ of complex numbers such that $c_i \neq 0$ for finitely many values of i [44]. Using this criterion it is straightforward to show that the first-order moments of a selfadjoint random matrix ensemble are the moments of a measure as long as they exist, i.e., as long as

$$\alpha_n = \lim_{N\to\infty} \mathbb{E}\left(N^{-1}\mathrm{Tr}\left(X_N^n\right)\right)$$

exists for every $n \in \mathbb{N}$. If, in addition, there exists $M > 0$ such that $|\alpha_n| \leq M^n$ for all $n \in \mathbb{N}$, then the underlying measure is compactly supported. This provides an integral representation for the (first-order) Cauchy transform, in the spirit of P2.

Recall that strong convergence in distribution amounts to the usual convergence of expectations plus the convergence of norms, see [28]. Over the years, many classical random matrix ensembles have been shown to converge strongly in distribution [18, 48, 9, 28, 1, 10, 4]. In connection with our last problem, note that $\mathbb{E}\left(N^{-1}\mathrm{Tr}\left((z\mathrm{I}_N - X_N)^{-1}\right)\right)$ equals

$$\mathbb{E}\left(N^{-1}\mathrm{Tr}\left((z\mathrm{I}_N - X_N)^{-1}\right)1_{\|X_N\|>M}\right) + \mathbb{E}\left(N^{-1}\mathrm{Tr}\left((z\mathrm{I}_N - X_N)^{-1}\right)1_{\|X_N\|\leq M}\right),$$

for any $M > 0$. If X_N converges strongly in distribution and M is large enough, then the first term in the previous equation tends to zero as N goes to infinity. Then a routine computation shows that if $|z|$ is large enough then the second term converges to

$$G(z) = \sum_{n \geq 0} \frac{\alpha_n}{z^{n+1}}.$$

In both computations the N^{-1} factor is crucial.

This discussion makes clear some of the difficulties in the second-order case. Although the linearization technique works also in the second-order level, at the moment we lack of an operator-valued version of second-order free probability theory. The classical criteria for the solvability of the moment problem in the plane [21] do not apply directly to our second problem. Specifically, the second-order moments are not the moments of the sought measure ν in formula (1.1). Finally, the N^{-1} present in the previous paragraph is absent in the second-order case. In particular, the above argument based on strong convergence in distribution does not imply that the covariance of (the traces of) resolvents converges to the second-order Cauchy transform.

The contributions of this thesis are as follows.

P1. In Chapter 3 we provide a formula for the second-order Cauchy transform of block Gaussian matrices in terms of their corresponding (matrix-valued) Cauchy transform. The proof of this formula relies on two main ingredients: the introduction of a second-order conditional expectation, and a systematic study of the single-line, double-line, and annular pairings and their relations.

P2. By studying two types of two-variables Cauchy transforms, their associated *moments* and inversion theorems, in Chapter 4 we establish a criterion, based on the positivity of a cluster function, that guarantees the veracity of the integral representation in (1.1). This approach is an alternative to the one by Huang and Mingo [24], where a Riesz-Herglotz formula in the plane is used.

P3. In Chapter 5 we prove that under some conditions the covariance of (the trace of) resolvents converges to the second-order Cauchy transform. These condition are: a weak form of a large deviation principle for the norm of a random matrix, and a Poincaré-type inequality for linear statistics. Moreover, these conditions also ensure the convergence of the covariance of analytic linear statistics to a certain contour integral depending on the second-order Cauchy transform. In addition, when the underlying random matrix ensemble has a second-order limit distribution, the same conditions lead to the asymptotic Gaussianity of continuously differentiable linear statistics, i.e., to the Central Limit Theorem for linear statistics. These results can be applied to many random matrix ensembles in the literature, including block Gaussian matrices.

Chapter 2

Preliminaries and Notation

In this chapter we gather some preliminary results and notation needed through this thesis. When appropriate, further references are provided.

2.1 General Notation

For $n \geq 1$, we let $[n] = \{1, \ldots, n\}$. Given $i'_1, \ldots, i'_m \in [d]$, we let $i' : [m] \to [d]$ be the function determined by $i'(s) = i'_s$. For $i' : [m] \to [d]$ and $i'' : [n] \to [d]$, we let $i : [m+n] \to [d]$ be the function given by

$$i(s) = i'_s 1_{s \leq m} + i''_{s-m} 1_{s>m},$$

where 1 is the indicator function. For notational simplicity, we use i_s instead of $i(s)$. Whenever d, m and n are clear from the context, we use the notation

$$\sum_{i,j} \text{ to represent the sum } \sum_{i'_1,\ldots,i'_m=1}^{d} \sum_{j'_1,\ldots,j'_m=1}^{d} \sum_{i''_1,\ldots,i''_n=1}^{d} \sum_{j''_1,\ldots,j''_n=1}^{d}.$$

For a given set $\mathcal{X}$, we let $\mathcal{X}^n = \mathcal{X} \times \cdots \times \mathcal{X}$ be its n-fold Cartesian product. We assume that all classical random variables belong to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with expectation $\mathbb{E}$. For random variables $X_1, \ldots, X_r$, we let $k_r(X_1, \ldots, X_r)$ denote their classical r-th cumulant, see [17, p. 185] for further details. In particular, for random variables X and Y, we have that $k_1(X) = \mathbb{E}(X)$ and $k_2(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

2.2 Partitions and Permutations

A partition of a non-empty set V is a family $\pi = \{V_1, \ldots, V_S\}$ such that $\emptyset \neq V_s \subset V$ for all $s \in [S]$, $V_s \cap V_t = \emptyset$ for $s \neq t$, and $\bigcup_{s \in [S]} V_s = V$. We say that π is a partition of n points if $V = [n]$. For notational convenience, we define $\emptyset$ as the only partition of 0 points. For an integer l and a partition $\pi = \{V_1, \ldots, V_S\}$ of a non-empty set $V \subset \mathbb{Z}$, we let $\pi + l$ be the partition of $\{v + l : v \in V\}$ given by

$$\pi + l = \big\{\{v + l : v \in V_s\} : s \in [S]\big\}.$$

We let $\emptyset + l := \emptyset$ for all $l \in \mathbb{Z}$. Let $\pi = \{V_1, \ldots, V_S\}$ be a partition of a non-empty set V; the relation $u \sim_\pi v$ whenever $u, v \in V_s$ for some $s \in [S]$ defines an equivalence relation on V; for $W \subset V$, we let $\pi|_W$ be the partition of W such that for every $u, v \in W$ we have that $u \sim_{\pi|_W} v$ if and only if $u \sim_\pi v$. A non-crossing partition of a non-empty set $V \subset \mathbb{Z}$ is a partition π of V such that if $a \sim_\pi b$, $c \sim_\pi d$, and $a < c < b < d$, then $a \sim_\pi b \sim_\pi c \sim_\pi d$. We denote by $\mathrm{NC}(n)$ the set of all non-crossing partitions of n points. We say that a partition $\pi = \{V_1, \ldots, V_S\}$ is a pairing if $|V_s| = 2$ for all $s \in [S]$. We denote by $\mathrm{NC}_2(n)$ the set of all non-crossing pairings of n points. By definition, the partition $\emptyset$ is a non-crossing pairing. Given a pairing of n points π, we let $\hat{\pi} = \{\{n+1-u, n+1-v\} : \{u, v\} \in \pi\}$. See [40, Lecture 9] for a more detailed discussion of non-crossing pairings.

Let $\mathcal{S}_V$ be the group of permutations of a non-empty set V. For a permutation π, let $\#(\pi)$ denote the number of cycles of π. For $m, n \in \mathbb{N}$, we define the permutation $\gamma_{m,n} : [m+n] \to [m+n]$ by

$$\gamma_{m,n}(p) = \begin{cases} p+1 & p \neq m \text{ and } p \neq m+n, \\ 1 & p = m, \\ m+1 & p = m+n. \end{cases}$$

By abuse of notation, given a pairing π of V, we let $\pi : V \to V$ be the permutation of V such that $\pi(u) = v$ whenever $v \neq u$ and $v \sim_\pi u$. A pairing of $m+n$ points π is called a (m,n)-annular non-crossing pairing if $\#(\gamma_{m,n}\pi) = \frac{m+n}{2}$ and there exists $u, v \in [m+n]$ such that $u \sim_\pi v$ and $u \leq m < v$. If we draw two concentric circles, the exterior one with m points labelled clockwise and the interior one with n points labelled counterclockwise, then the (m,n)-annular non-crossing pairings correspond to the pairings of these points that can be drawn without crossings and that have at least one string connecting both circles. Figure 2.1 depicts the $(4,4)$-annular non-crossing pairing $\pi = \{\{1,6\},\{2,5\},\{3,4\},\{7,8\}\}$. We denote by $\mathrm{NC}_2(m,n)$ the set of all (m,n)-annular non-crossing pairings. See [35, Sec. 5.1] for a detailed discussion about non-crossing annular permutations.

2.3 Matrices

Let $\mathrm{M}_{m\times n}(\mathcal{A})$ denote the set of all $m \times n$ matrices with entries in $\mathcal{A}$. We let A^* and A^{T} denote the conjugate transpose and transpose of a matrix A, respectively. For

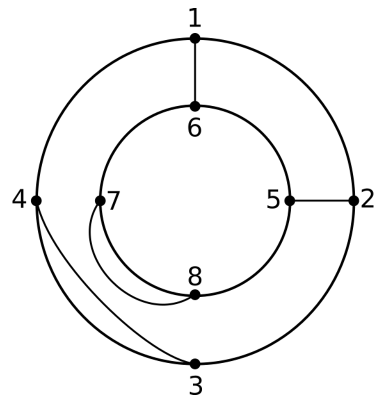

Figure 2.1: Graphical representation of a $(4,4)$-annular non-crossing pairing.

notational simplicity, we write $\mathrm{M}_n(\mathcal{A})$ instead of $\mathrm{M}_{n\times n}(\mathcal{A})$. When $\mathcal{A} = \mathbb{C}$, we simply write $\mathrm{M}_{m\times n}$ and M_n. For a matrix $A \in \mathrm{M}_{m\times n}(\mathcal{A})$, we let $A(p,q)$ be the p,q-entry of A. We denote by $\mathrm{Tr} : \mathrm{M}_n(\mathcal{A}) \to \mathcal{A}$ the trace function given by $\mathrm{Tr}(A) = \sum_{k=1}^{n} A(k,k)$. Also, we let $\|A\|$ be the operator norm of $A \in \mathrm{M}_n$,

$$\|A\| = \sup\{\|Ax\|_2 : x \in \mathbb{C}^n, \|x\|_2 \leq 1\},$$

where $\|x\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$. For $A \in \mathrm{M}_{m'\times n'}$ and $B \in \mathrm{M}_{m''\times n''}$, we let $A \otimes B$ be the Kronecker product of A and B, i.e., the $m'm'' \times n'n''$ matrix given by

$$A \otimes B = \begin{pmatrix} A(1,1)B & A(1,2)B & \cdots & A(1,n')B \\ A(2,1)B & A(2,2)B & \cdots & A(2,n')B \\ \vdots & \vdots & \ddots & \vdots \\ A(m',1)B & A(m',2)B & \cdots & A(m',n')B \end{pmatrix}.$$

For $X \in \mathrm{M}_{d^2}$, we let $X(p,q;r,s) = X((p-1)d+r,(q-1)d+s)$ for all $p,q,r,s \in [d]$. In particular, if $A, B \in \mathrm{M}_d$, then

$$(A \otimes B)(p,q;r,s) = A(p,q)B(r,s). \tag{2.1}$$

With this notation, if $X, Y \in \mathrm{M}_{d^2}$ then, for $p,q,r,s \in [d]$,

$$(XY)(p,q;r,s) = \sum_{k,l=1}^{d} X(p,k;r,l)Y(k,q;l,s). \tag{2.2}$$

Assume that $X_N^{(p,q)}$ is an $N \times N$ (random) matrix for each $p, q \in [d]$. We let $X_N = (X_N^{(p,q)})_{p,q}$ be the $dN \times dN$ (random) matrix given by

$$X_N = \begin{pmatrix} X_N^{(1,1)} & X_N^{(1,2)} & \cdots & X_N^{(1,d)} \\ X_N^{(2,1)} & X_N^{(2,2)} & \cdots & X_N^{(2,d)} \\ \vdots & \vdots & \ddots & \vdots \\ X_N^{(d,1)} & X_N^{(d,2)} & \cdots & X_N^{(d,d)} \end{pmatrix}. \tag{2.3}$$

In particular, $X_N(p,q;r,s) = X_N^{(p,q)}(r,s)$. Identifying $\mathrm{M}_{dN} \cong \mathrm{M}_d \otimes \mathrm{M}_N$, we let

$$(\mathbf{1}_{\mathrm{M}_d} \otimes \mathrm{Tr}_{\mathrm{M}_N})(X_N) = \left(\mathrm{Tr}\left(X_N^{(p,q)}\right)\right)_{p,q=1}^{d} \in \mathrm{M}_d$$

where $\mathbf{1}_{\mathrm{M}_d}$ and $\mathrm{Tr}_{\mathrm{M}_N}$ are the identity on M_d and the trace on M_N, respectively. When there is no risk of confusion, we write $\mathbf{1} \otimes \mathrm{Tr}$ without any reference to the specific spaces.

2.4 Second-Order Free Probability

The following definitions are a selection of the key concepts of second-order free probability theory required in this thesis. Note that no mention to second-order freeness is made. The reason is that this thesis is mainly concerned with a single random matrix ensemble, thus no interaction between matrices is present. Further details about the notion of second-order freeness can be found in the series of papers [36, 34, 12].

A second-order non-commutative probability space $(\mathcal{A}, \varphi, \rho)$ consists of a unital algebra $\mathcal{A}$, a tracial linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(1) = 1$, and a bilinear functional $\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ which is tracial in both arguments and satisfies $\rho(x, 1) = \rho(1, x) = 0$ for all $x \in \mathcal{A}$. Given a non-commutative random variable $x \in \mathcal{A}$, its second-order Cauchy transform G_2 is defined as the moment generating function

$$G_2(z, w) = \sum_{m,n \geq 0} \frac{\rho(x^m, x^n)}{z^{m+1} w^{n+1}}.$$

For non-commuting random variables $x_1, \ldots, x_N \in \mathcal{A}$ and $i_1, \ldots, i_m \in [N]$, we let $\prod_{w=1}^{m} x_{i_w}$ denote

$$x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_m}$$

in that specific order.

Definition 1. Let $(\mathcal{A}, \varphi, \rho)$ be a second-order probability space. We say that a family of selfadjoint operators $x_1, \ldots, x_N \in \mathcal{A}$ is a second-order semicircular family with covariance $\sigma : [N] \times [N] \to \mathbb{C}$ if for all $m, n \in \mathbb{N}$, $i' : [m] \to [N]$ and $i'' : [n] \to [N]$ we have that

$$\varphi\left(\prod_{w=1}^{m} x_{i'_w}\right) = \sum_{\pi\in \mathrm{NC}_2(m)} \prod_{u \underset{\pi}{\sim} v} \sigma(i'_u; i'_v),$$

$$\rho\left(\prod_{w=1}^{m} x_{i'_w}, \prod_{w=1}^{n} x_{i''_w}\right) = \sum_{\pi\in \mathrm{NC}_2(m,n)} \prod_{u \underset{\pi}{\sim} v} \sigma(i_u; i_v). \tag{2.4}$$

The following definitions are closely related to the global fluctuations of random matrices.

Definition 2. Let $(X_N : N \in \mathbb{N})$ be a random matrix ensemble. We say that it has a second-order limit distribution if

i) For all $m, n \in \mathbb{N}$, the following limits exist

$$\alpha_n = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\left(X_N^n\right)\right) \quad \text{and} \quad \alpha_{m,n} = \lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left(X_N^m\right), \mathrm{Tr}\left(X_N^n\right)\right),$$

where $\mathrm{Cov}\left(X, Y\right) = \mathbb{E}\left(XY\right) - \mathbb{E}\left(X\right)\mathbb{E}\left(Y\right)$;

ii) For all $r \geq 3$ and all $n_1, \ldots, n_r \in \mathbb{N}$,

$$\lim_{N\to\infty} k_r\left(\mathrm{Tr}\left(X_N^{n_1}\right), \ldots, \mathrm{Tr}\left(X_N^{n_r}\right)\right) = 0,$$

where k_r denotes the classical cumulant of order r.

Definition 3. For each $N \in \mathbb{N}$, let $\{A_N^{(1)}, \ldots, A_N^{(S)}\}$ be a collection of $N \times N$ random matrices. Assume that $(\mathcal{A}, \varphi, \rho)$ is a second-order probability space and $a_1, \ldots, a_S$ are non-commutative random variables in $\mathcal{A}$. We say that $\{A_N^{(1)}, \ldots, A_N^{(S)}\}$ converges in second-order distribution to $\{a_1, \ldots, a_S\}$, denoted by $\{A_N^{(1)}, \ldots, A_N^{(S)}\} \overset{\text{so-dist}}{\longrightarrow} \{a_1, \ldots, a_S\}$, if for all polynomials $P_1, P_2, \ldots$ in S non-commuting indeterminates we have that

$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\mathrm{Tr}\left[P_1\big(A_N^{(1)},\ldots,A_N^{(S)}\big)\right]\right) = \varphi\left(p_1\right),$$

$$\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left[P_1\big(A_N^{(1)},\ldots,A_N^{(S)}\big)\right], \mathrm{Tr}\left[P_2\big(A_N^{(1)},\ldots,A_N^{(S)}\big)\right]\right) = \rho(p_1,p_2),$$

where $p_k = P_k(a_1,\ldots,a_S)$ for all $k \in \mathbb{N}$, and, for $r \geq 3$,

$$\lim_{N\to\infty} k_r\left(\mathrm{Tr}\left[P_1\big(A_N^{(1)},\ldots,A_N^{(S)}\big)\right], \ldots, \mathrm{Tr}\left[P_r\big(A_N^{(1)},\ldots,A_N^{(S)}\big)\right]\right) = 0.$$

Chapter 3

Second-Order Cauchy Transform of Block Gaussian Matrices

In the following section we present the main result of this chapter, namely, a formula for the second-order Cauchy transform associated to block Gaussian matrices. The matricial analogues of the second-order moments, Cauchy transform, and semicircular elements are introduced in Section 3.2. In Section 3.3 we establish the connection between the block Gaussian matrices and the matricial second-order semicircular elements. In Section 3.4 we establish the properties about single-line, double-line, and annular pairings required to establish, in Section 3.5, the combinatorial expressions for the Cauchy transforms associated to these pairings. Finally, in Section 3.6, we extend these formulas to an analytic level.

3.1 Setting and Main Results

Let $d \in \mathbb{N}$ be fixed and let $\sigma : [d]^2 \times [d]^2 \to \mathbb{C}$ be a given covariance mapping. A selfadjoint $dN \times dN$ random matrix X_N is called a block Gaussian matrix with covariance σ if

$$X_N = \begin{pmatrix} X_N^{(1,1)} & X_N^{(1,2)} & \cdots & X_N^{(1,d)} \\ \vdots & \vdots & \ddots & \vdots \\ X_N^{(d,1)} & X_N^{(d,2)} & \cdots & X_N^{(d,d)} \end{pmatrix},$$

where $\{X_N^{(p,q)} : p, q \in [d]\}$ are $N \times N$ selfadjoint random matrices such that

$$\{\Re(X_N^{(p,q)}(i,j)), \Im(X_N^{(p,q)}(i,j)) : p, q \in [d], i, j \in [N]\}$$

are jointly Gaussian random variables with zero mean and covariance specified by

$$\mathbb{E}\left(X_N^{(p,q)}(k',l')X_N^{(r,s)}(k'',l'')\right) = \frac{1}{N}\delta_{k',l''}\delta_{l',k''}\sigma(p,q;r,s).$$

Note that the Gaussian Unitary Ensemble (GUE) corresponds to the case $d = 1$. For each $n \in \mathbb{N}$, let $\alpha_n = \lim_{N\to\infty} N^{-1}\mathbb{E}\left(\mathrm{Tr}\left(X_N^n\right)\right)$. These first-order moments are encoded in the generating function

$$G(z) = \sum_{n\geq 0} \frac{\alpha_n}{z^{n+1}}. \tag{3.1}$$

We call this generating function the Cauchy transform of the first-order moments of X_N, however here we are not concerned with its representation as the Cauchy transform of a measure. Using a conditional expectation taking values over the $d \times d$ complex matrices, Helton et al. [22] proved the following. Let $\eta : \mathrm{M}_d \to \mathrm{M}_d$ be given by $\eta(w)_{p,q} = \sum_{k,l} w_{k,l}\sigma(p,k;l,q)$. For a given $z \in \mathbb{C}$ with $\Im(z) > 0$, let $T_z : \mathrm{M}_d \to \mathrm{M}_d$ be determined by $T_z(w) = (z\mathrm{I}_d - \eta(w))^{-1}$. If $|z|$ is large enough, then the right hand side of (3.1) converges absolutely and

$$G(z) = d^{-1}\mathrm{Tr}\left(\mathcal{G}(z)\right), \tag{3.2}$$

where $\mathcal{G} : \mathbb{C}^+ \to \mathrm{M}_d$ is an analytic function given by

$$\mathcal{G}(z) = \lim_{n\to\infty} T_z^{\circ n}(w) \tag{3.3}$$

for any $w \in \mathrm{M}_d$ with $\Im w < 0$. Note that $\mathcal{G}(z) = T_z(\mathcal{G}(z))$.

In this chapter we extend the previous analysis to the global fluctuations of block Gaussian matrices. Specifically, for each $m, n \in \mathbb{N}$, let

$$\alpha_{m,n} = \lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left(X_N^m\right), \mathrm{Tr}\left(X_N^n\right)\right).$$

These second-order moments are encoded in the generating function

$$G_2(z,w) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{\alpha_{m,n}}{z^{m+1}w^{n+1}}. \tag{3.4}$$

As before, we call this generating function the second-order Cauchy transform of X_N. In the next chapter, we address the question of whether it can be represented as a two-variable Cauchy transform of a measure on the plane. The main result of this chapter provides a closed form expression for G_2 in terms of the mapping $\mathcal{G}$. Let $\Sigma \in \mathrm{M}_d \otimes \mathrm{M}_d$ be the matrix given by $\Sigma(p,q;r,s) = \sigma(p,q;r,s)$, cf. Section 2.3. Also, for every $A \in \mathrm{M}_d$ let $\Theta(A)(p,q;r,s) = A(p,r;q,s)$, $A^\Gamma(p,q;r,s) = A(p,q;s,r)$, and $\Phi(A) = \Theta(A^\Gamma)$.

Theorem 1. Let X_N be a block Gaussian matrix with covariance σ. There exists $M \in \mathbb{R}_+$ such that if $z, w \in \mathbb{C}$ with $|z|, |w| > M$, then the right hand side of (3.4) converges absolutely and

$$G_2(z,w) = \mathrm{Tr}\left(\mathcal{G}_2(z,w)\right),$$

where $\mathcal{G}_2 : (\mathbb{C}\backslash[-M,M])^2 \to \mathrm{M}_{d^2}$ is an analytic function such that $\mathcal{G}_2(z,w)$ equals

$$\Theta\left(\mathcal{G}_D(z,z)\left\{\Theta[\mathcal{H}(z,w)] + \Phi\left[\mathcal{H}(z,w)^{\Gamma}(\mathcal{G}(z)\otimes\mathcal{G}(w))^{\Gamma}\mathcal{H}(z,w)^{\Gamma}\right]\right\}\mathcal{G}_D(w,w)^{\mathrm{T}}\right) \quad (3.5)$$

with

$$\mathcal{H}(z,w) = \left(\mathrm{I}_{d^2} - \Sigma[\mathcal{G}(z)\otimes\mathcal{G}(w)]\right)^{-1}\Sigma,$$
$$\mathcal{G}_D(z,w) = [\mathcal{G}(z)\otimes\mathcal{G}(w)]\left(\mathrm{I}_{d^2} - \Sigma[\mathcal{G}(z)\otimes\mathcal{G}(w)]\right)^{-1}.$$

Proof. This theorem is a direct consequence of Theorem 3 and Corollary 2 below. □

When $d = 1$ and $\sigma(1,1;1,1) = 1$, (3.5) becomes the much simpler equation

$$G_2(z,w) = \frac{G(z)^2 G(w)^2}{[1-G(z)^2][1-G(w)^2][1-G(z)G(w)]^2}.$$

Since the Cauchy transform of the semicircle distribution satisfies that

$$-zG'(z)[1-G(z)^2] = G(z)^2,$$

we obtain that

$$G_2(z,w) = \frac{zwG'(z)G'(w)}{[1-G(z)G(w)]^2}.$$

This equation can be found in an unpublished work of Mingo and Nica. Thus, our main theorem can be regarded as a generalization of this formula beyond the case $d = 1$. Another important feature of Mingo and Nica's formula is that G_2 can be computed from G. Similarly, (3.5) readily shows that $\mathcal{G}_2$ can be obtained from $\mathcal{G}$.

3.2 MSO Moments and the MSO Cauchy Transform

Assume that $(\mathcal{A}, \varphi, \rho)$ is a second-order non-commutative probability space. Let $\mathrm{M}_d(\mathcal{A})$ be the algebra of $d \times d$ matrices over $\mathcal{A}$. A natural conditional expectation $E : \mathrm{M}_d(\mathcal{A}) \to \mathrm{M}_d$ is given by

$$E(X)(p, q) = \varphi(X(p, q)),$$

for all $X \in \mathrm{M}_d(\mathcal{A})$ and $p, q \in [d]$. Motived by the previous equation, in this section we introduce a *conditional* version of ρ which leads to the notions of matricial second-order moments and matricial second-order Cauchy transform. As shown in the following section, they can be used to obtain the second-order moments and Cauchy transforms of certain block random matrices. In particular, the second-order behavior of the block Gaussian matrices introduced in Section 3.1 can be obtained from the matricial second-order semicircular elements defined below. The main result of this section establishes that the matricial second-order Cauchy transform associated to matricial second-order semicircular elements admits a description in terms of the non-crossing annular pairings.

Recall that ρ is bilinear on $\mathcal{A} \times \mathcal{A}$, so it can be naturally extended to $\mathcal{A} \otimes \mathcal{A}$. We define $P : \mathrm{M}_d(\mathcal{A}) \otimes \mathrm{M}_d(\mathcal{A}) \to \mathrm{M}_d \otimes \mathrm{M}_d$ by

$$P(X \otimes Y) = (\mathbf{1}_{\mathrm{M}_d \otimes \mathrm{M}_d} \otimes \rho)(X \otimes Y),$$

for all $X, Y \in \mathrm{M}_d(\mathcal{A})$. In the previous equation, we identify $\mathrm{M}_d(\mathcal{A}) \otimes \mathrm{M}_d(\mathcal{A})$ with $(\mathrm{M}_d \otimes \mathrm{M}_d) \otimes (\mathcal{A} \otimes \mathcal{A})$. In the convention from Section 2.3, the previous equation reads as

$$P(X \otimes Y)(p, q; r, s) = \rho(X(p, q), Y(r, s)),$$

for all $p, q, r, s \in [d]$. Note that $P(X \otimes Y)$ is a $d^2 \times d^2$ matrix, in contrast to $E(X)$ which is a $d \times d$ matrix. Observe that, for all $a \in \mathrm{M}_d$ and all $X \in \mathrm{M}_d(\mathcal{A})$,

$$P(a \otimes X) = P(X \otimes a) = 0. \tag{3.6}$$

Also, since ρ is bilinear, equations (2.1) and (2.2) readily imply that

$$P(aXb \otimes cYd) = (a \otimes c)P(X \otimes Y)(b \otimes d) \tag{3.7}$$

for all $a, b, c, d \in \mathrm{M}_d$ and $X, Y \in \mathrm{M}_d(\mathcal{A})$. Features (3.6) and (3.7) make P a second-order analogue of the conditional expectation E: $E(a) = a$ and $E(aXb) = aE(X)b$ for all $a, b \in \mathrm{M}_d$ and $X \in \mathrm{M}_d(\mathcal{A})$.

Definition 4. Let $X \in \mathrm{M}_d(\mathcal{A})$. We define the matricial second-order (MSO) (m, n)-moment of X by

$$M_{m,n}(\mathbf{a}; \mathbf{b}) = P(a_0 X a_1 \cdots X a_m \otimes b_0 X b_1 \cdots X b_n)$$

where $\mathbf{a} = (a_0, \ldots, a_m) \in \mathrm{M}_d^{m+1}$ and $\mathbf{b} = (b_0, \ldots, b_n) \in \mathrm{M}_d^{n+1}$. By abuse of notation, we let $M_{m,n}(a; b)$ denote $M_{m,n}(a, \ldots, a; b, \ldots, b)$ for every $a, b \in \mathrm{M}_d$.

We define the matricial second-order (MSO) Cauchy transform of X by

$$\mathcal{G}_2(a, b) = \sum_{m,n \geq 1} M_{m,n}(a^{-1}; b^{-1}) \tag{3.8}$$

for $a, b \in \mathrm{M}_d$ invertible.

Equation (3.8) should be interpreted at the level of formal expressions. In other words, we think of $\mathcal{G}_2(a, b)$ as an element of

$$\mathrm{M}_{d^2}\left(\mathbb{C}[[\{a(i,j), b(i,j) : i, j \in [d]\}]]\right),$$

the algebra of $d^2 \times d^2$ matrices over the formal power series in the $2d^2$ commuting variables $\{a(i,j), b(i,j) : i, j \in [d]\}$. We only evaluate (3.8) whenever a and b are invertible and the series (3.8) converges. In Section 3.6 we explore some analytical properties of $\mathcal{G}_2(a, b)$ when $X \in \mathrm{M}_d(\mathcal{A})$ is a matricial second-order semicircular element.

Definition 5. We say that $X \in \mathrm{M}_d(\mathcal{A})$ is a matricial second-order (MSO) semicircular element with covariance $\sigma : [d]^2 \times [d]^2 \to \mathbb{C}$ if

i) $X(p, q) = X(q, p)$ for all $p, q \in [d]$;

ii) $\{X(p, q) \mid p, q \in [d]\}$ is a second-order semicircular family with covariance σ (Definition 1).

Observe that by i), for all $p, q, r, s \in [d]$,

$$\sigma(p, q; r, s) = \sigma(p, q; s, r) = \sigma(q, p; r, s) = \sigma(q, p; s, r). \tag{3.9}$$

The next proposition establishes a moment-cumulant-like formula for MSO semicircular elements. For $i' : [m] \to [d]$ and $i'' : [n] \to [d]$, we let $i : [m+n] \to [d]$ be given by $i(s) = i'_s 1_{s \leq m} + i''_{s-m} 1_{s>m}$. Also, recall that $\sum\limits_{i,j}$ denotes the sum $\sum\limits_{i'_1,\ldots,i'_m} \sum\limits_{j'_1,\ldots,j'_m} \sum\limits_{i''_1,\ldots,i''_n} \sum\limits_{j''_1,\ldots,j''_n}$ where the indices run from 1 to d.

Proposition 1. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. Then

$$M_{m,n}(\mathbf{a};\mathbf{b}) = \sum_{\pi \in \mathrm{NC}_2(m,n)} \kappa_\pi(\mathbf{a};\mathbf{b})$$

where

$$\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s) = \sum_{i,j} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=0}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v) \tag{3.10}$$

with $j'_0 := p$, $i'_{m+1} := q$, $j''_0 := s$, and $i''_{n+1} := r$.

The apparently unnecessary transpose introduced in (3.10) will simplify the expression for the double-line Cauchy transform introduced in Section 3.5.2. Observe that κ_π depends on X only through σ.

Proof. By i) in Definition 5, we have that $X = X^T$. In particular, $(b_0 X \cdots X b_n)^{\mathrm{T}} = b_n^{\mathrm{T}} X \cdots X b_0^{\mathrm{T}}$ and hence

$$M_{m,n}(\mathbf{a};\mathbf{b})(p,q;r,s) = P(a_0 X \cdots X a_m \otimes b_n^{\mathrm{T}} X \cdots X b_0^{\mathrm{T}})(p,q;s,r). \tag{3.11}$$

A straightforward computation leads to

$$\begin{aligned}(a_0 X \cdots X a_m)(p,q) &= \sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} a_0(j'_0, i'_1) X(i'_1, j'_1) \cdots X(i'_m, j'_m) a_m(j'_m, i'_{m+1}) \\ &= \sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \prod_{w=1}^{m} X(i'_w, j'_w),\end{aligned} \tag{3.12}$$

with $j'_0 := p$ and $i'_{m+1} := q$. Similarly, we have that

$$(b_n^{\mathrm{T}} X \cdots X b_0^{\mathrm{T}})(s,r) = \sum_{i''_1,\ldots,i''_n} \sum_{j''_1,\ldots,j''_n} \left[\prod_{w=0}^{n} b_{n-w}^{\mathrm{T}}(j''_w, i''_{w+1})\right] \prod_{w=1}^{n} X(i''_w, j''_w), \tag{3.13}$$

with $j''_0 = s$ and $i''_{n+1} = r$. Since ρ is bilinear, equations (3.11), (3.12), and (3.13) imply that $M_{m,n}(\mathbf{a};\mathbf{b})(p,q;r,s)$ equals

$$\sum_{i,j} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=0}^{n} b_{n-w}^{\mathrm{T}}(j''_w, i''_{w+1})\right] \rho\left(\prod_{w=1}^{m} X(i'_w, j'_w), \prod_{w=1}^{n} X(i''_w, j''_w)\right).$$

By ii) in Definition 5, the entries of X form a second-order semicircular family. Equation (2.4) implies then that $M_{m,n}(\mathbf{a};\mathbf{b})(p,q;r,s)$ equals

$$\sum_{\pi\in \mathrm{NC}_2(m,n)} \sum_{i,j} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=0}^{n} b_{n-w}^{\mathrm{T}}(j''_w, i''_{w+1})\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v),$$

as required. □

By abuse of notation, we let $\kappa_\pi(a;b)$ denote $\kappa_\pi(a,\ldots,a;b,\ldots,b)$ for every $a,b \in \mathrm{M}_d$

.

Corollary 1. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. Then its MSO Cauchy transform is given by

$$\mathcal{G}_2(a,b) = \sum_{m,n\geq 1} \sum_{\pi\in \mathrm{NC}_2(m,n)} \kappa_\pi(a^{-1}; b^{-1}).$$

In Section 3.5 we establish a recursive description for κ_π which allows us to obtain an explicit expression for $\mathcal{G}_2$ in terms of $\mathcal{G}$. In the following section we connect the matricial second-order objects introduced here with the corresponding second-order objects associated to block random matrices.

3.3 Fluctuations of Block Random Matrices

We start this section connecting $P : \mathrm{M}_d(\mathcal{A}) \otimes \mathrm{M}_d(\mathcal{A}) \to \mathrm{M}_d \otimes \mathrm{M}_d$ as introduced in the previous section with the covariance of traces of block random matrices. Recall the notion of convergence in second-order distribution from Definition 3.

Proposition 2. For each $N \in \mathbb{N}$, let $\{X_N^{(p,q)}, Y_N^{(p,q)} | p, q \in [d]\}$ be $N \times N$ random matrices. Consider the $dN \times dN$ block random matrices $X_N = (X_N^{(p,q)})_{p,q}$ and $Y_N = (Y_N^{(p,q)})_{p,q}$. Assume that

$$\{X_N^{(p,q)}, Y_N^{(p,q)} | p, q \in [d]\} \xrightarrow{\text{so-dist}} \{X(p,q), Y(p,q) | p, q \in [d]\}$$

in some second-order probability space $(\mathcal{A}, \varphi, \rho)$ where $X, Y \in \mathrm{M}_d(\mathcal{A})$. Then

$$\lim_{N \to \infty} k_2 (\mathrm{Tr}(X_N), \mathrm{Tr}(Y_N)) = \mathrm{Tr}(P(X \otimes Y)).$$

Proof. For each $N \in \mathbb{N}$, let $A_N = (\mathbf{1} \otimes \mathrm{Tr})(X_N)$ and $B_N = (\mathbf{1} \otimes \mathrm{Tr})(Y_N)$. For all $N \in \mathbb{N}$, both A_N and B_N are $d \times d$ matrices with $A_N(p,q) = \mathrm{Tr}(X_N^{(p,q)})$ and $B_N(p,q) = \mathrm{Tr}(Y_N^{(p,q)})$ for all $p, q \in [d]$. Since the trace of the tensor product of two matrices equals the product of the traces of the individual matrices, we have that

$$\begin{aligned} k_2(\mathrm{Tr}(X_N), \mathrm{Tr}(Y_N)) &= k_2(\mathrm{Tr}(A_N), \mathrm{Tr}(B_N)) \\ &= \mathbb{E}(\mathrm{Tr}(A_N)\,\mathrm{Tr}(B_N)) - \mathbb{E}(\mathrm{Tr}(A_N))\,\mathbb{E}(\mathrm{Tr}(B_N)) \end{aligned}$$

$$= \mathrm{Tr}\left(\mathbb{E}\left(A_N \otimes B_N\right) - \mathbb{E}\left(A_N\right) \otimes \mathbb{E}\left(B_N\right)\right). \tag{3.14}$$

Let $C_N = \mathbb{E}\left(A_N \otimes B_N\right) - \mathbb{E}\left(A_N\right) \otimes \mathbb{E}\left(B_N\right)$. Observe that, for all $p, q, r, s \in [d]$,

$$\begin{aligned} C_N(p,q;r,s) &= \mathbb{E}\left(\mathrm{Tr}\left(X_N^{(p,q)}\right)\mathrm{Tr}\left(Y_N^{(r,s)}\right)\right) - \mathbb{E}\left(\mathrm{Tr}\left(X_N^{(p,q)}\right)\right)\mathbb{E}\left(\mathrm{Tr}\left(Y_N^{(r,s)}\right)\right) \\ &= k_2\left(\mathrm{Tr}\left(X_N^{(p,q)}\right), \mathrm{Tr}\left(Y_N^{(r,s)}\right)\right). \end{aligned}$$

By the assumed convergence in second-order distribution, the previous equation implies that

$$\lim_{N\to\infty} C_N(p,q;r,s) = \rho(X(p,q), Y(r,s)) = P(X \otimes Y)(p,q;r,s).$$

Plugging the previous limit in (3.14), we obtain that $\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left(X_N\right), \mathrm{Tr}\left(Y_N\right)\right) = \mathrm{Tr}\left(P(X \otimes Y)\right)$. □

The following proposition relates the second-order Cauchy transform and its matricial counterpart. It is important to observe the similarity between the formulas in (3.2) and (3.15).

Proposition 3. Let $X_N = (X_N^{(p,q)})_{p,q}$ be a $dN \times dN$ block random matrix such that

$$\{X_N^{(p,q)} : p, q \in [d]\} \overset{\text{so-dist}}{\longrightarrow} \{X(p,q) : p, q \in [d]\}$$

in some second-order probability space $(\mathcal{A}, \varphi, \rho)$ where $X \in \mathrm{M}_d(\mathcal{A})$. For $m, n \in \mathbb{N}$, $\mathbf{a} \in \mathrm{M}_d^{m+1}$, and $\mathbf{b} \in \mathrm{M}_d^{n+1}$, let

$$A_N = (a_0 \otimes \mathrm{I}_N)X_N \cdots X_N(a_m \otimes \mathrm{I}_N) \quad \text{and} \quad B_N = (b_0 \otimes \mathrm{I}_N)X_N \cdots X_N(b_n \otimes \mathrm{I}_N).$$

Similarly, let $A = a_0 X \cdots X a_m$ and $B = b_0 X \cdots X b_n$. Then,

$$\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left(A_N\right), \mathrm{Tr}\left(B_N\right)\right) = \mathrm{Tr}\left(P(A \otimes B)\right).$$

In particular, we have that

$$G_2(z,w) = \mathrm{Tr}\left(\mathcal{G}_2(z\mathrm{I}_d, w\mathrm{I}_d)\right). \tag{3.15}$$

Proof. Using the convention adopted in (2.3), we have that

$$\begin{aligned}
A_N^{(p,q)} &= \sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} (a_0 \otimes \mathrm{I}_N)^{(p,i'_1)} X_N^{(i'_1,j'_1)} \cdots X_N^{(i'_n,j'_n)} (a_m \otimes \mathrm{I}_N)^{(j'_n,q)} \\
&= \sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} (a_0(p,i'_1)\mathrm{I}_N) X_N^{(i'_1,j'_1)} \cdots X_N^{(i'_n,j'_n)} (a_m(j'_n,q)\mathrm{I}_N) \\
&= \sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \prod_{w=1}^{m} X_N^{(i'_w,j'_w)},
\end{aligned}$$

where $j'_0 := p$ and $i'_{n+1} := q$. Similarly,

$$B_N^{(r,s)} = \sum_{i''_1,\ldots,i''_n} \sum_{j''_1,\ldots,j''_n} \left[\prod_{w=0}^{n} b_w(j''_w, i''_{w+1})\right] \prod_{w=1}^{n} X_N^{(i''_w,j''_w)},$$

where $j''_0 := r$ and $i''_{n+1} := s$. A straightforward computation shows that

$$\begin{aligned}
A(p,q) &= \sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \prod_{w=1}^{m} X(i'_w, j'_w), \\
B(r,s) &= \sum_{i''_1,\ldots,i''_n} \sum_{j''_1,\ldots,j''_n} \left[\prod_{w=0}^{n} b_w(j''_w, i''_{w+1})\right] \prod_{w=1}^{n} X(i''_w, j''_w),
\end{aligned}$$

where $j'_0 := p$, $i'_{n+1} := q$, $j''_0 := r$, and $i''_{n+1} := s$. The assumed convergence in second-order distribution immediately implies that $\{A_N^{(p,q)}, B_N^{(p,q)} | p, q \in [d]\}$ converges in second-order distribution to $\{A(p,q), B(p,q) | p, q \in [d]\}$. By the previous proposition,

$$\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left(A_N\right), \mathrm{Tr}\left(B_N\right)\right) = \mathrm{Tr}\left(P(A \otimes B)\right).$$

In particular, we have that $\alpha_{m,n} := \lim_{N\to\infty} k_2(\mathrm{Tr}\left(X_N^m\right), \mathrm{Tr}\left(X_N^n\right)) = \mathrm{Tr}\left(P(X^m \otimes X^n)\right)$ and

$$\begin{aligned} G_2(z,w) &= \sum_{m,n\geq 0} z^{-(m+1)} w^{-(n+1)} \alpha_{m,n} \\ &= \sum_{m,n\geq 0} \mathrm{Tr}\left(P(z^{-1}X\cdots Xz^{-1} \otimes w^{-1}X\cdots Xw^{-1})\right) \\ &= \mathrm{Tr}\left(\mathcal{G}_2(z\mathrm{I}_d, w\mathrm{I}_d)\right), \end{aligned}$$

as required. □

The following is a restatement of Theorem 3.1 [36] in our notation.

Theorem 2. Let $X_N = (X_N^{(p,q)})_{p,q}$ be a selfadjoint $dN \times dN$ random matrix such that $\{X_N^{(p,q)} | p, q \in [d]\}$ are $N \times N$ selfadjoint random matrices such that

$$\{\Re(X_N^{(p,q)}(i,j)), \Im(X_N^{(p,q)}(i,j)) : p, q \in [d], i, j \in [N]\}$$

are jointly Gaussian random variables with zero mean and covariance specified by

$$\mathbb{E}\left(X_N^{(p,q)}(k',l') X_N^{(r,s)}(k'',l'')\right) = \frac{1}{N}\delta_{k',l''}\delta_{l',k''}\sigma(p,q;r,s),$$

where $\sigma : [d]^2 \times [d]^2 \to \mathbb{C}$ is a given covariance mapping. If $X \in \mathrm{M}_d(\mathcal{A})$ is a MSO semicircular element with covariance σ, then

$$\{X_N^{(p,q)} : p, q \in [d]\} \xrightarrow{\text{so-dist}} \{X(p,q) : p, q \in [d]\}.$$

Hence, by the previous theorem and (3.15), the second-order Cauchy transform of a block Gaussian matrix can be obtained from the MSO Cauchy transform of the limiting MSO semicircular element. In what follows we focus on computing the latter.

3.4 Single-Line, Double-Line and Annular Pairings

In this section we consider three types of pairings: single-line, double-line, and annular. These pairings, and their relations, play an important role in the computation of the MSO Cauchy transform of MSO semicircular elements. The combinatorial facet of free probability theory centers in the notion of non-crossing partitions. Whenever we talk about partitions (pairings), we implicitly refer to non-crossing partitions (pairings).

3.4.1 Single-Line Pairings

Let $\mathrm{NC}_2(n)$ be the set of non-crossing pairings of n points. In the following section we introduce a type of pairing called double-line. To make a clear distinction between the different types of pairings, we refer to the usual non-crossing pairings as *single-line pairings*. For uniformity of notation, we let $\mathrm{SP}_n := \mathrm{NC}_2(n)$ for all $n \geq 1$ and $\mathrm{SP}_0 = \{\emptyset\}$. The set of all single-line pairings SP is then given by

$$\mathrm{SP} = \bigcup_{n \geq 0} \mathrm{SP}_n.$$

A word about the name single-line pairing is in order. The nesting of the blocks of a given partition is not particularly important when computing its corresponding scalar-valued free cumulant. However, this nesting is crucial in the operator-valued setting. This feature is well known in the literature [50], but for concreteness consider the following.

Example 1. For a semicircular element $x \in \mathcal{A}$, its scalar-valued free cumulants satisfy

$$\kappa_{\pi_1}^x = \varphi\left(x^2\right)^n = \kappa_{\pi_2}^x$$

for all $\pi_1, \pi_2 \in \mathrm{SP}_{2n}$. Let $\pi_1 = \{\{1,2\},\{3,4\}\}$ and $\pi_2 = \{\{1,4\},\{2,3\}\}$. For an M_d-valued semicircular element X with covariance σ, its M_d-valued free cumulants satisfy

$$\kappa_{\pi_1}^X(a,a,a,a,a) = a\eta(a)a\eta(a)a \quad \text{and} \quad \kappa_{\pi_2}^X(a,a,a,a,a) = a\eta(a\eta(a)a)a$$

for all $a \in \mathrm{M}_d$ where $\eta : \mathrm{M}_d \to \mathrm{M}_d$ is given by $\eta(a)(p,q) = \sum_{k,l} a(k,l)\sigma(p,k;l,q)$. In general, we have that $\kappa_{\pi_1}^X(a,a,a,a,a) \neq \kappa_{\pi_2}^X(a,a,a,a,a)$.

For notational convenience, in this thesis we think of κ_π, with $\pi \in \mathrm{SP}_{2n}$, as a $2n+1$-linear map. In the usual convention, e.g., Section 9.1 in [35], this cumulant is actually a $2n-1$-linear map. Note that these two conventions differ by a scalar matrix a at the beginning and the end of the cumulant, as seen in the previous example.

In scalar-valued free probability theory it is customary to represent the elements of $\mathrm{NC}(n)$ as non-crossing partitions of n points in the circle. In this graphical representation the nested structure of non-crossing partitions is somehow buried. As recalled in the previous example, this nesting is crucial in the operator-valued setting. To emphasize this feature, we represent the elements of SP_n as (non-crossing) pairings of n points located along a single line, as depicted in Figure 3.1.

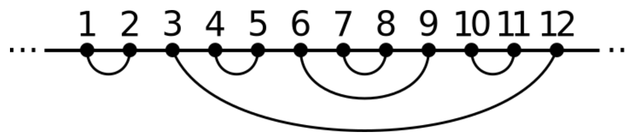

Figure 3.1: Graphical representation of a single-line pairing.

3.4.2 Double-Line Pairings

In this section we introduce a type of pairing called double-line. As the name suggests, these pairings may be represented as pairings of points located along two parallel lines.

Definition 6. For $m, n \geq 0$, we define the set of double-line pairings of m and n points $\mathrm{DP}_{m,n}$ as

$$\mathrm{DP}_{m,n} := \{(m, n, \pi) : \pi \in \mathrm{NC}_2(m+n)\}.$$

When there is no risk of confusion, we write $\pi \in \mathrm{DP}_{m,n}$ instead of $(m, n, \pi) \in \mathrm{DP}_{m,n}$. For $\pi \in \mathrm{DP}_{m,n}$, a pair $\{u, v\} \in \pi$ is called a through string if either $u \leq m < v$ or $v \leq m < u$. Let $\mathrm{DP}_{m,n}^{(k)}$ be the set of double-line pairings of m and n points with exactly k through strings. The set of all double-line pairings with exactly k through-strings and the set of all double pairings are then given by

$$\mathrm{DP}^{(k)} = \bigcup_{m,n \geq 0} \mathrm{DP}_{m,n}^{(k)} \quad \text{and} \quad \mathrm{DP} = \bigcup_{m,n \geq 0} \mathrm{DP}_{m,n}.$$

Note that if $m_1 \neq m_2$, and $m_1 + n_1 = m_2 + n_2$, and $\pi \in \mathrm{NC}_2(m_1 + n_1)$, then (m_1, n_1, π) and (m_2, n_2, π) are different elements of DP. A word about the name double-line pairing is in order. We think of the elements of $\mathrm{DP}_{m,n}$ as non-crossing pairings of points located along two parallel lines, one with m points labelled in increasing order and the other with n points labelled in decreasing order. In this context, a through string is a pair that connects one line to the other.

Example 2. Consider the non-crossing pairing

$$\pi = \{\{1,2\},\{3,12\},\{4,5\},\{6,9\},\{7,8\},\{10,11\}\} \in \mathrm{NC}_2(12).$$

A graphical representation of π is provided in Figure 3.1. The double-line pairings $\pi_1 = (4,8,\pi) \in \mathrm{DP}_{4,8}$ and $\pi_2 = (6,6,\pi) \in \mathrm{DP}_{6,6}$ are depicted in Figure 3.2.

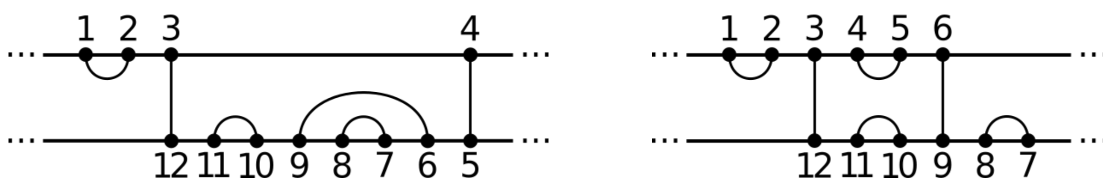

Figure 3.2: Examples of double-line pairings.

The following proposition, whose proof is a routine verification, provides a useful relation between single-line and double-line pairings. Recall that for a given set $\mathcal{X}$, we let $\mathcal{X}^n = \mathcal{X} \times \cdots \times \mathcal{X}$ be its n-fold Cartesian product.

Proposition 4. For $k \geq 0$, consider the function $\Psi^{(k)} : \mathrm{SP}^{k+1} \times \mathrm{SP}^{k+1} \to \mathrm{DP}^{(k)}$ such that $\Psi^{(k)}(\pi'_0, \ldots, \pi'_k; \pi''_k, \ldots, \pi''_0)$ equals

$$\bigcup_{i=0}^{k} \left(\pi'_i + i + \sum_{j=0}^{i} m_j - m_i \right) \cup \bigcup_{i=0}^{k} \left(\pi''_{k-i} + m + i + \sum_{j=0}^{i} n_{k-j} - n_{k-i} \right) \cup$$
$$\cup \left\{ \left\{ i + \sum_{j=0}^{i-1} m_j, m + n - \left(i - 1 + \sum_{j=0}^{i-1} n_j \right) \right\} : i \in [k] \right\} \in \mathrm{DP}^{(k)}_{m,n},$$

where $\pi'_j \in \mathrm{SP}_{m_j}$ and $\pi''_j \in \mathrm{SP}_{n_j}$ for all $0 \leq j \leq k$, $m = k + \sum_{j=0}^{k} m_j$, and $n = k + \sum_{j=0}^{k} n_j$. The mapping $\Psi^{(k)}$ is a bijection between $\mathrm{SP}^{k+1} \times \mathrm{SP}^{k+1}$ and $\mathrm{DP}^{(k)}$.

Naturally, there exists a bijection between $\mathrm{SP}^{k+1} \times \mathrm{SP}^{k+1}$ and $\mathrm{DP}^{(k)}$ as both sets are countably infinite. The importance of the previous proposition comes from the fact that the mapping $\Psi^{(k)}$ allows us to express the cumulants of double-line pairings in terms of the cumulants of single-line pairings, as shown in Section 3.5.2.

In graphical terms, $\Psi^{(k)}$ glues the single-line pairings $\pi'_0, \ldots, \pi'_k, \pi''_k, \ldots, \pi''_0$ successively and intertwines them with through strings, see Figure 3.3. Note that π'_i and π''_i are in front of each other. E.g., if π_1 and π_2 are the double-line pairings in Example 2, then

$$\pi_1 = \Psi^{(2)}(\{\{1,2\}\}, \emptyset, \emptyset; \emptyset, \{\{1,4\}, \{2,3\}, \{5,6\}\}, \emptyset),$$
$$\pi_2 = \Psi^{(2)}(\{\{1,2\}\}, \{\{1,2\}\}, \emptyset; \{\{1,2\}\}, \{\{1,2\}\}, \emptyset).$$

Observe that in the double-line representation of $\Psi^{(k)}(\pi'_0, \ldots, \pi'_k; \pi''_k, \ldots, \pi''_0)$ the depictions of $\pi''_k, \ldots, \pi''_0$ are graphically reversed.

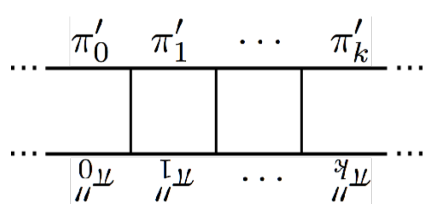

Figure 3.3: Graphical representation of $\Psi^{(k)}(\pi'_0, \ldots, \pi'_k; \pi''_k, \ldots, \pi''_0)$.

A family of double-line pairings that plays an important role in the computation of the MSO Cauchy transform of MSO semicircular elements is

$$\mathrm{DP}^{||} := \bigcup_{m,n\geq 1} \{\pi \in \mathrm{DP}_{m,n} : \{1, m+n\}, \{m, m+1\} \in \pi\}\,.$$

Note that there is a very natural bijection between DP and $\mathrm{DP}^{||}\backslash\mathrm{DP}_{1,1}$ which is implemented as follows. For a given $m, n \in \mathbb{N}$, let $h : \mathbb{N} \to \mathbb{N}$ be given by $h(u) = 1 + u + 2 \cdot 1_{u>m}$. Then, the mapping

$$\mathrm{DP}_{m,n} \ni \pi \mapsto \{\{1, m+n+4\}, \{m+2, m+3\}\} \cup \{\{h(u), h(v)\} : \{u, v\} \in \pi\} \in \mathrm{DP}_{m+2,n+2} \tag{3.16}$$

is a bijection between $\mathrm{DP}_{m,n}$ and $\mathrm{DP}_{m+2,n+2} \cap \mathrm{DP}^{||}$. A graphical representation of the action of this mapping is shown in Figure 3.4.

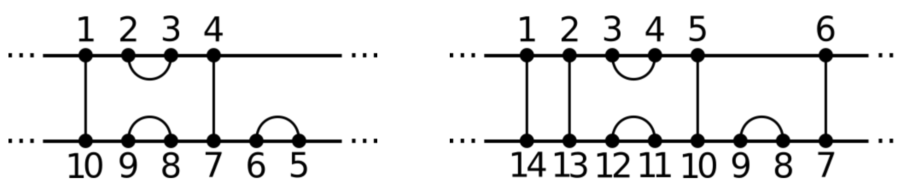

Figure 3.4: A double-line pairing and its image under the isomorphism in (3.16).

3.4.3 Annular Pairings

Let $\mathrm{NC}_2(m, n)$ be the set of non-crossing annular pairings of m exterior and n interior points. For uniformity of notation, we let $\mathrm{AP}_{m,n} := \mathrm{NC}_2(m, n)$ for all $m, n \geq 1$. For $\pi \in \mathrm{AP}_{m,n}$, a pair $\{u, v\} \in \pi$ is called a through string if either $u \leq m < v$ or $v \leq m < u$. Let $\mathrm{AP}^{(k)}_{m,n}$ be the set of annular pairings of m exterior and n interior points with exactly k through strings. The set of all annular pairings with exactly k

through strings and the set of all annular pairings are then given by

$$\mathrm{AP}^{(k)} = \bigcup_{m,n\geq 1} \mathrm{AP}^{(k)}_{m,n} \quad \text{and} \quad \mathrm{AP} = \bigcup_{m,n\geq 1} \mathrm{AP}_{m,n}.$$

Recall that, by definition, a pairing in $\mathrm{AP}_{m,n}$ must have at least one through string. Aiming for a simpler computation of the MSO Cauchy transform of MSO semicircular elements, we divide the annular pairings into two classes.

Definition 7. Let $\pi \in \mathrm{AP}^{(k)}_{m,n}$. Let $1 \leq t'_1 < t'_2 < \cdots < t'_k \leq m$ and $1 \leq t''_1, \ldots, t''_k \leq n$ be such that $\{t'_1, m + t''_1\}, \ldots, \{t'_k, m + t''_k\}$ are the through strings of π. We say that π is of Type I, denoted by $\pi \in \mathcal{T}_{\mathrm{I}}$, if $t''_1 \geq t''_k$. If $t''_1 < t''_k$, we say that π is of Type II, and denote it by $\pi \in \mathcal{T}_{\mathrm{II}}$.

As it is customary, [31, 36], we represent the elements of $\mathrm{AP}_{m,n}$ as non-crossing pairings of points located along two concentric circles, see Figure 2.1. In this work we make special emphasis on the starting and ending points of the circles. This distinction, which is immaterial in the scalar-valued setting, is crucial in the matrix-valued computations of the following section. Graphically, Type I annular pairings have their exterior and interior circles aligned with respect to where they start and end, while Type II annular pairings are somehow shifted, see Figure 3.5.

Type I Annular Pairings

Type I annular pairings can be decomposed into three double-line pairings as follows.

Definition 8. Let $\pi \in \mathrm{AP}^{(k)}_{m,n} \cap \mathcal{T}_{\mathrm{I}}$ and denote by $\{t'_1, m + t''_1\}, \ldots, \{t'_k, m + t''_k\}$ its through strings with $t'_1 < \cdots < t'_k$ (so $t''_1 > \cdots > t''_k$). We define

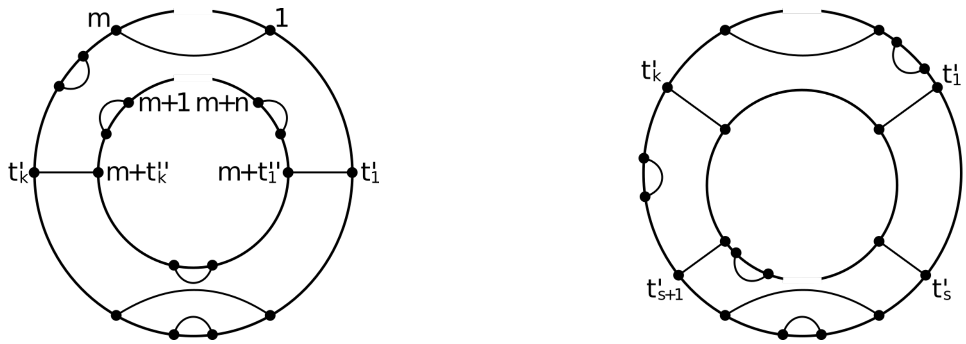

Figure 3.5: Example of a Type I and a Type II annular pairing.

a) the exterior boundary of π, $\pi_{eb} := \pi|_{[1,t'_1)\cup(t'_k,m]}$;

b) the interior boundary of π, $\pi_{ib} := \pi|_{[m+1,m+t''_k)\cup(m+t''_1,m+n]}$;

c) the regular part of π, $\pi_{re} := \pi|_{[t'_1,t'_k]\cup[m+t''_k,m+t''_1]}$.

In Figure 3.6 there is a graphical representation of the above decomposition for the Type I annular pairing in Figure 3.5. Observe that we can identify π_{eb} with an element in $\mathrm{DP}_{t'_1-1,m-t'_k}$. Specifically, we can identify π_{eb} with

$$\{\{h(u),h(v)\}:\{u,v\}\in\pi_{eb}\}\in\mathrm{DP}_{t'_1-1,m-t'_k}, \tag{3.17}$$

where $h:\mathbb{N}\to\mathbb{N}$ is given by $h(u)=u1_{u<t'_1}+(u-t'_k+t'_1-1)1_{u\geq t'_1}$. Similarly, we can identify π_{ib} with an element in $\mathrm{DP}_{t''_k-1,n-t''_1}$. Also, we can identify π_{re} with the element of $\mathrm{DP}^{||}$ given by

$$\{\{h(u),h(v)\}:\{u,v\}\in\pi_{re}\}\in\mathrm{DP}_{t'_k-t'_1+1,t''_1-t''_k+1}, \tag{3.18}$$

where $h:\mathbb{N}\to\mathbb{N}$ is given by $h(u)=(u-t'_1+1)1_{u\leq m}+(u-m-t''_k+1+t'_k-t'_1+1)1_{u>m}$. For ease of notation, we denote by π_{eb} both π_{eb} and its associated element in DP;

similarly for π_{ib} and π_{re}. The identifications implemented in (3.17) and (3.18) readily imply the following.

Proposition 5. The mapping

$$\mathcal{T}_{\mathrm{I}} \ni \pi \mapsto (\pi_{eb}, \pi_{ib}, \pi_{re}) \in \mathrm{DP} \times \mathrm{DP} \times \mathrm{DP}^{||} \tag{3.19}$$

is a bijection between $\mathcal{T}_{\mathrm{I}}$ and $\mathrm{DP} \times \mathrm{DP} \times \mathrm{DP}^{||}$.

As in Proposition 4, the existence of a bijection between $\mathcal{T}_{\mathrm{I}}$ and $\mathrm{DP} \times \mathrm{DP} \times \mathrm{DP}^{||}$ is obvious. The relevance of the previous proposition comes from the fact that the bijection in (3.19) allows us to express the cumulants of Type I annular pairings in terms of the cumulants of double-line pairings, as shown in Section 3.5.3.

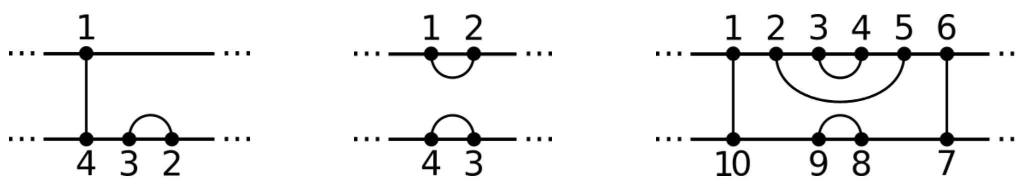

Figure 3.6: Decomposition of a Type I annular pairing: π_{eb}, π_{ib}, and π_{re}.

Type II Annular Pairings

As we did for Type I annular pairings, we decompose Type II annular pairings into four double-line pairings and two single-line pairings. Even though this decomposition may seem complicated, it is very useful to compute the cumulants of Type II annular pairings, as shown in the following section.

Definition 9. Let $\pi \in \mathrm{AP}_{m,n}^{(k)} \cap \mathcal{T}_{\mathrm{II}}$ and denote by $\{t_1', m + t_1''\}, \ldots, \{t_k', m + t_k''\}$ its through strings with $t_1' < \cdots < t_k'$and $t_{s+1}'' > \cdots > t_k'' > t_1'' > \cdots > t_s''$ for some $s \in [k]$.

We define

a) the exterior boundary of π, $\pi_{eb} := \pi|_{[1,t'_1)\cup(t'_k,m]}$;

b) the interior boundary of π, $\pi_{ib} := \pi|_{[m+1,m+t''_s)\cup(m+t''_{s+1},m+n]}$;

c) the right regular part of π, $\pi_{rr} := \pi|_{[t'_1,t'_s]\cup[m+t''_s,m+t''_1]}$;

d) the left regular part of π, $\pi_{lr} := \pi|_{[t'_{s+1},t'_k]\cup[m+t''_k,m+t''_{s+1}]}$;

e) the oposite part to the interior boundary of π, $\pi_{oi} := \pi|_{(t'_s,t'_{s+1})}$;

f) the oposite part to the exterior boundary of π, $\pi_{oe} := \pi|_{(m+t''_1,m+t''_k)}$.

In Figure 3.7 there is a graphical representation of the above decomposition for the Type II annular pairing in Figure 3.5. As we did before, we can identify each of the six pieces of the previous decomposition with either a single-line or a double-line pairing. In this case, these identifications lead to the following.

Proposition 6. The mapping

$$\mathcal{T}_{\mathrm{II}} \ni \pi \mapsto (\pi_{eb}, \pi_{ib}, \pi_{rr}, \pi_{lr}, \pi_{oi}, \pi_{oe}) \in \mathrm{DP} \times \mathrm{DP} \times \mathrm{DP}^{||} \times \mathrm{DP}^{||} \times \mathrm{SP} \times \mathrm{SP}$$

is a bijection between $\mathcal{T}_{\mathrm{II}}$ and $\mathrm{DP} \times \mathrm{DP} \times \mathrm{DP}^{||} \times \mathrm{DP}^{||} \times \mathrm{SP} \times \mathrm{SP}$.

3.5 Cauchy Transforms of MSO Semicircular Elements

In this section we define and compute the MSO Cauchy transforms associated to single-line, double-line, and annular pairings.

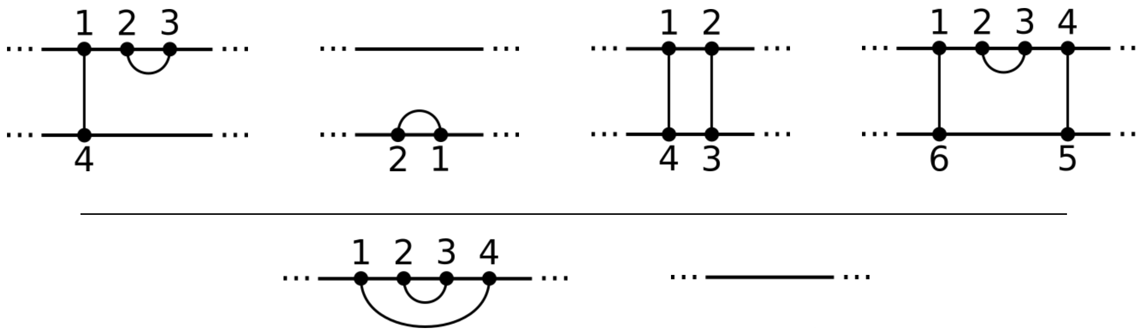

Figure 3.7: Decomposition of a Type II annular pairing: π_{eb}, π_{ib}, π_{rr}, π_{lr}, π_{oi}, and π_{oe}.

3.5.1 Single-Line Pairings

Recall that single-line pairings are the usual non-crossing pairings, i.e., $\mathrm{SP}_n = \mathrm{NC}_2(n)$ for $n \geq 1$ and $\mathrm{SP}_0 = \{\emptyset\}$. We adopt the operator-valued free cumulants as the cumulants associated to single-line pairings. Specifically, if $X \in \mathrm{M}_d(\mathcal{A})$ is a MSO semicircular element with covariance σ, the cumulant $\kappa_\pi : \mathrm{M}_d^{n+1} \to \mathrm{M}_d$ associated to $\pi \in \mathrm{SP}_n$ is given by

$$\kappa_\pi(a_0, \ldots, a_n)(p, q) = \sum_{i_1,\ldots,i_n} \sum_{j_1,\ldots,j_n} \left[\prod_{w=0}^{n} a_w(j_w, i_{w+1}) \right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v),$$

where $j_0 := p$ and $i_{n+1} := q$. Recall that [50], for all $(a_0, \ldots, a_n) \in \mathrm{M}_d^{n+1}$,

$$E(a_0 X \cdots X a_n) = \sum_{\pi \in \mathrm{SP}_n} \kappa_\pi(a_0, \ldots, a_n). \tag{3.20}$$

In particular, the cumulants associated to single-line pairings satisfy a moment-cumulant formula.

Single-line cumulants obey a recursive computation rule that depends on the nested structure of the underlying pairings [50]. Even though we won't rely explicitly on this property, for completeness consider the following.

Example 3. Let $\pi \in \mathrm{SP}_{12}$ be the non-crossing pairing given in Figure 3.1. In this case, for $\mathbf{a} \in \mathrm{M}_d^{13}$,

$$\kappa_\pi(\mathbf{a}) = a_0\eta(a_1)a_2\eta(a_3\eta(a_4)a_5\eta(a_6\eta(a_7)a_8)a_9\eta(a_{10})a_{11})a_{12},$$

where $\eta : \mathrm{M}_d \to \mathrm{M}_d$ is given by $\eta(a)(p,q) = \sum_{k,l} a(k,l)\sigma(p,k;l,q)$.

The following lemma will be useful later in this section.

Lemma 1. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. If $\pi \in \mathrm{SP}_n$, then

$$\kappa_\pi(a_n^{\mathrm{T}}, \ldots, a_0^{\mathrm{T}})^{\mathrm{T}} = \kappa_{\hat{\pi}}(a_0, \ldots, a_n)$$

for all $(a_0, \ldots, a_n) \in \mathrm{M}_d^{n+1}$ where $\hat{\pi} = \{\{n+1-u, n+1-v\} : \{u,v\} \in \pi\}$.

Proof. Let $p, q \in [d]$. By definition of κ_π, we have that

$$\begin{aligned}\kappa_\pi(a_n^{\mathrm{T}}, \ldots, a_0^{\mathrm{T}})(q,p) &= \sum_{i_1,\ldots,i_n} \sum_{j_1,\ldots,j_n} \left[\prod_{w=0}^{n} a_{n-w}^{\mathrm{T}}(j_w, i_{w+1})\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v) \\ &= \sum_{i_1,\ldots,i_n} \sum_{j_1,\ldots,j_n} \left[\prod_{w=0}^{n} a_{n-w}(i_{w+1}, j_w)\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v),\end{aligned}$$

with $j_0 := q$ and $i_{n+1} := p$. Making the change of variables $i_w \to j'_{n+1-w}$ and $j_w \to i'_{n+1-w}$, so $j'_0 = p$ and $i'_{n+1} = q$, we obtain that $\kappa_\pi(a_n^{\mathrm{T}}, \ldots, a_0^{\mathrm{T}})(q,p)$ equals

$$\sum_{i'_1,\ldots,i'_n} \sum_{j'_1,\ldots,j'_n} \left[\prod_{w=0}^{n} a_{n-w}(j'_{n-w}, i'_{n+1-w})\right] \prod_{u \underset{\pi}{\sim} v} \sigma(j'_{n+1-u}, i'_{n+1-u}; j'_{n+1-v}, i'_{n+1-v}),$$

which equals

$$\sum_{i'_1,\ldots,i'_n}\sum_{j'_1,\ldots,j'_n}\left[\prod_{w=0}^{n} a_w(j'_w, i'_{w+1})\right]\prod_{u\underset{\pi}{\sim}v}\sigma(j'_{n+1-u}, i'_{n+1-u}; j'_{n+1-v}, i'_{n+1-v}).$$

The change of variables $n+1-u \to u$ and $n+1-v \to v$ leads to

$$\prod_{u\underset{\pi}{\sim}v}\sigma(j'_{n+1-u}, i'_{n+1-u}; j'_{n+1-v}, i'_{n+1-v}) = \prod_{n+1-u\underset{\pi}{\sim}n+1-v}\sigma(j'_u, i'_u; j'_v, i'_v).$$

By definition of $\hat{\pi}$, we have that $\{n+1-u, n+1-v\} \in \pi$ if and only if $\{u, v\} \in \hat{\pi}$. Therefore,

$$\kappa_\pi(a_n^{\mathrm{T}}, \ldots, a_0^{\mathrm{T}})(q,p) = \sum_{i'_1,\ldots,i'_n}\sum_{j'_1,\ldots,j'_n}\left[\prod_{w=0}^{n} a_w(j'_w, i'_{w+1})\right]\prod_{u\underset{\hat{\pi}}{\sim}v}\sigma(j'_u, i'_u; j'_v, i'_v).$$

By definition of a MSO semicircular element, the covariance mapping σ satisfies (3.9). In particular, $\kappa_\pi(a_n^{\mathrm{T}}, \ldots, a_0^{\mathrm{T}})(q,p)$ equals

$$\sum_{i'_1,\ldots,i'_n}\sum_{j'_1,\ldots,j'_n}\left[\prod_{w=0}^{n} a_w(j'_w, i'_{w+1})\right]\prod_{u\underset{\hat{\pi}}{\sim}v}\sigma(i'_u, j'_u; i'_v, j'_v) = \kappa_{\hat{\pi}}(a_0, \ldots, a_n)(p,q).$$

Since this equation holds for every $p, q \in [d]$, we conclude that

$$\kappa_\pi(a_n^{\mathrm{T}}, \ldots, a_0^{\mathrm{T}})^{\mathrm{T}} = \kappa_{\hat{\pi}}(a_0, \ldots, a_n),$$

as required. □

By abuse of notation, we let $\kappa_\pi(a) := \kappa_\pi(a, \ldots, a)$.

Definition 10. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. We define the single-line Cauchy transform $G_S : \mathrm{M}_d \to \mathrm{M}_d$ of X by

$$G_S(a) = \sum_{\pi \in \mathrm{SP}} \kappa_\pi(a^{-1}).$$

The purpose of the subindex S in G_S is to distinguish this Cauchy transform, that depends only on the single-line pairings, from the Cauchy transforms in the following sections. Note that G_S is a particular case of the operator-valued Cauchy transform in [50]. It is important to remark that G_S is a formal expression version of the analytic mapping $\mathcal{G}$ considered in the introduction. Since $\mathcal{G}$ can be easily computed using the fixed point equation (3.3), we use the single-line Cauchy transform as the basic building block for the upcoming Cauchy transforms.

3.5.2 Double-Line Pairings

In this section we define the cumulants associated to the double-line pairings and relate them to the cumulants of the single-line pairings. For a given covariance mapping $\sigma : [d]^2 \times [d]^2 \to \mathbb{C}$, we let Σ be the $d^2 \times d^2$ matrix determined by $\Sigma(p,q;r,s) = \sigma(p,q;r,s)$. Given $i' : [m] \to [d]$ and $i'' : [n] \to [d]$, we let $i : [m+n] \to [d]$ be the function $i_p = i'_p 1_{p \le m} + i''_{p-m} 1_{p>m}$. Recall that $\sum_{i,j}$ denotes the sum $\sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} \sum_{i''_1,\ldots,i''_n} \sum_{j''_1,\ldots,j''_n}$ where the indices run from 1 to d.

Definition 11. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. We define the cumulant $\kappa_\pi : \mathrm{M}_d^{m+1} \times \mathrm{M}_d^{n+1} \to \mathrm{M}_{d^2}$ associated to $\pi \in \mathrm{DP}_{m,n}$ by

$$\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s) = \sum_{i,j} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1}) \right] \left[\prod_{w=0}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1}) \right] \prod_{\substack{u \sim v \\ \pi}} \sigma(i_u, j_u; i_v, j_v),$$

with $j'_0 := p$, $i'_{m+1} := q$, $j''_0 := s$, and $i''_{n+1} := r$.

Despite the fact that these functions are called cumulants, in principle, they do not satisfy a moment-cumulant formula like the one in (3.20). Nonetheless, these cumulants play an important role in the computation of the MSO Cauchy transform of MSO semicircular elements. The following lemma will be useful later in this section.

Lemma 2. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. If $\pi \in \mathrm{DP}_{m,n}$, then

$$\kappa_\pi(a_0, \ldots, a_m; b_0, \ldots, b_n)(p, q; r, s) = \kappa_{\hat{\pi}}(b_0, \ldots, b_n; a_0, \ldots, a_m)(r, s; p, q)$$

where $\hat{\pi} = \{\{m + n + 1 - u, m + n + 1 - v\} : \{u, v\} \in \pi\} \in \mathrm{DP}_{n,m}$.

Proof. By definition, we have that

$$\kappa_\pi(\mathbf{a}; \mathbf{b})(p, q; r, s) = \sum_{i,j} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=0}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v),$$

with $j'_0 := p$, $i'_{m+1} := q$, $j''_0 := s$, and $i''_{n+1} := r$. Recall that, by assumption, the covariance mapping σ satisfies (3.9). A straightforward computation shows that

$$\kappa_\pi(\mathbf{a}; \mathbf{b})(p, q; r, s) = \sum_{i,j} \left[\prod_{w=0}^{n} b_{n-w}(i''_{w+1}, j''_w)\right] \left[\prod_{w=0}^{m} a^{\mathrm{T}}_w(i'_{w+1}, j'_w)\right] \prod_{u \underset{\pi}{\sim} v} \sigma(j_u, i_u; j_v, i_v).$$

Consider the change of variables

$$i''_w \to \hat{j}'_{n+1-w} \text{ and } j''_w \to \hat{i}'_{n+1-w} \text{ for all } w \in [n],$$

$$i'_w \to \hat{j}''_{m+1-w} \text{ and } j'_w \to \hat{i}''_{m+1-w} \text{ for all } w \in [m],$$

with $\hat{j}'_0 := r$, $\hat{i}'_{n+1} := s$, $\hat{j}''_0 := q$, and $\hat{i}''_{m+1} := p$. Observe that

$$i_p = i'_p 1_{p\leq m} + i''_{p-m} 1_{p>m} = \hat{j}''_{m+1-p} 1_{n+1\leq m+n+1-p} + \hat{j}'_{m+n+1-p} 1_{n+1>m+n+1-p} = \hat{j}_{m+n+1-p}.$$

Similarly, we have that $j_p = \hat{i}_{m+n+1-p}$. Thus, $\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s)$ equals

$$\sum_{\hat{i},\hat{j}} \left[\prod_{w=0}^{n} b_{n-w}(\hat{j}'_{n-w}, \hat{i}'_{n-w+1})\right] \left[\prod_{w=0}^{m} a_w^{\mathrm{T}}(\hat{j}''_{m-w}, \hat{i}''_{m-w+1})\right] \times$$
$$\times \prod_{u \underset{\pi}{\sim} v} \sigma(\hat{i}_{m+n+1-u}, \hat{j}_{m+n+1-u}; \hat{i}_{m+n+1-v}, \hat{j}_{m+n+1-v}).$$

Since $\{u,v\} \in \pi$ if and only if $\{m+n+1-u, m+n+1-v\} \in \hat{\pi}$, the change of variables $m+n+1-u \to u$ and $m+n+1-v \to v$ leads to

$$\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s) = \sum_{\hat{i},\hat{j}} \left[\prod_{w=0}^{n} b_w(\hat{j}'_w, \hat{i}'_{w+1})\right] \left[\prod_{w=0}^{m} a_{m-w}^{\mathrm{T}}(\hat{j}''_w, \hat{i}''_{w+1})\right] \prod_{u \underset{\hat{\pi}}{\sim} v} \sigma(\hat{i}_u, \hat{j}_u; \hat{i}_v, \hat{j}_v).$$

Observe that the right hand side of the previous equation equals

$$\kappa_{\hat{\pi}}(b_0, \ldots, b_n; a_0, \ldots, a_m)(r,s;p,q).$$

The result follows. □

In the following lemma we express the cumulant associated to a double-line pairing in terms of the cumulants of single-line pairings. Let $\Psi^{(k)} : \mathrm{SP}^{k+1} \times \mathrm{SP}^{k+1} \to \mathrm{DP}^{(k)}$ be the bijection introduced in Proposition 4.

Lemma 3. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. If $\pi \in \mathrm{DP}^{(k)}_{m,n}$ has through strings $\{t'_1, m+t''_1\}, \ldots, \{t'_k, m+t''_k\}$ with $t'_1 < \cdots < t'_k$ (so $t''_1 > \cdots > t''_k$), then $\kappa_\pi(\mathbf{a};\mathbf{b})$ equals

$$\left[\kappa_{\pi'_0}(a_0, \ldots, a_{t'_1-1}) \otimes \kappa_{\widehat{\pi''_0}}(b_0, \ldots, b_{n-t''_1})\right] \Sigma \cdots \Sigma \left[\kappa_{\pi'_k}(a_{t'_k}, \ldots, a_m) \otimes \kappa_{\widehat{\pi''_k}}(b_{n-t''_k+1}, \ldots, b_n)\right],$$

where $\pi = \Psi^{(k)}(\pi'_0, \ldots, \pi'_k; \pi''_k, \ldots, \pi''_0)$.

To illustrate the previous lemma, consider the following.

Example 4. Let π_1 and π_2 be the double-line pairings in Example 2. For appropriate $\mathbf{a}$ and $\mathbf{b}$,

$$\kappa_{\pi_1}(\mathbf{a};\mathbf{b}) = [(a_0\eta(a_1)a_2) \otimes b_0]\Sigma[a_3 \otimes (b_1\eta(b_2)b_3\eta(b_4\eta(b_5)b_6)b_7)]\Sigma[a_4 \otimes b_8],$$
$$\kappa_{\pi_2}(\mathbf{a};\mathbf{b}) = [(a_0\eta(a_1)a_2) \otimes b_0]\Sigma[(a_3\eta(a_4)a_5) \otimes (b_1\eta(b_2)b_3)]\Sigma[a_6 \otimes (b_4\eta(b_5)b_6)].$$

Proof of Lemma 3. We proof this lemma by induction on $k \geq 0$. If $\pi \in \mathrm{DP}^{(0)}_{m,n}$, Proposition 4 implies that $\pi = \Psi^{(0)}(\pi'_0; \pi''_0)$ where π'_0 and π''_0 are pairings of $[1,m]$ and $[1,n]$, respectively. In particular,

$$\prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v) = \prod_{u \underset{\pi'_0}{\sim} v} \sigma(i'_u, j'_u; i'_v, j'_v) \times \prod_{u \underset{\pi''_0}{\sim} v} \sigma(i''_u, j''_u; i''_v, j''_v)$$

for every $i', j' : [m] \to [d]$ and $i'', j'' : [n] \to d$. Fix $p, q, r, s \in [d]$. The previous equation implies that

$$\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s) = \sum_{i'_1,\ldots,i'_m} \sum_{j'_1,\ldots,j'_m} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \prod_{u \underset{\pi'_0}{\sim} v} \sigma(i'_u, j'_u; i'_v, j'_v) \times$$

$$\times \sum_{i''_1,\ldots,i''_n} \sum_{j''_1,\ldots,j''_n} \left[\prod_{w=0}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{\substack{u \sim v \\ \pi''_0}} \sigma(i''_u, j''_u; i''_v, j''_v),$$

with $j'_0 := p$, $i'_{m+1} := q$, $j''_0 := s$ and $i''_{n+1} := r$. The right-hand side of the previous equation is the product the cumulants of two single-line pairings. Specifically,

$$\kappa_\pi(\mathbf{a}; \mathbf{b})(p, q; r, s) = \kappa_{\pi'_0}(a_0, \ldots, a_n)(p, q)\kappa_{\pi''_0}(b^{\mathrm{T}}_n, \ldots, b^{\mathrm{T}}_0)(s, r).$$

By Lemma 1, we have that $\kappa_\pi(\mathbf{a}; \mathbf{b})(p, q; r, s)$ equals

$$\kappa_{\pi'_0}(a_0, \ldots, a_n)(p, q)\kappa_{\widehat{\pi''_0}}(b_0, \ldots, b_n)(r, s) = \left[\kappa_{\pi'_0}(\mathbf{a}) \otimes \kappa_{\widehat{\pi''_0}}(\mathbf{b})\right](p, q; r, s).$$

Since the last equality holds for all $p, q, r, s \in [d]$, we conclude that $\kappa_\pi(\mathbf{a}, \mathbf{b}) = \kappa_{\pi'_0}(\mathbf{a}) \otimes \kappa_{\widehat{\pi''_0}}(\mathbf{b})$.

Assume that the lemma is true for $k-1$. Let $\pi \in \mathrm{DP}^{(k)}_{m,n}$ be as in the statement of the lemma. By definition of $\Psi^{(k)}$, we have that

$$\pi = \pi'_0 \cup (\pi''_0 + m + t''_1) \cup \{t'_1, m + t''_1\} \cup \tilde{\pi},$$

where $\tilde{\pi} = \pi|_{(t'_1, m+t''_1)}$ is a pairing of $\{t'_1 + 1, \ldots, m + t''_1 - 1\}$. Recall that, by assumption, the covariance mapping σ satisfies (3.9). The previous equality implies that $\prod_{\substack{u \sim v \\ \pi}} \sigma(i_u, j_u; i_v, j_v)$ equals

$$\sigma(i'_{t'_1}, j'_{t'_1}; j''_{t''_1}, i''_{t''_1}) \prod_{\substack{u \sim v \\ \pi'_0}} \sigma(i'_u, j'_u; i'_v, j'_v) \prod_{\substack{u \sim v \\ \pi''_0 + m + t''_1}} \sigma(i''_{u-m}, j''_{u-m}; i''_{v-m}, j''_{v-m}) \prod_{\substack{u \sim v \\ \tilde{\pi}}} \sigma(i_u, j_u; i_v, j_v),$$

and therefore

$$\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s) = \sum_{i'_{t'_1}, j'_{t'_1}, i''_{t''_1}, j''_{t''_1}} A(i'_{t'_1}, j''_{t''_1})\Sigma(i'_{t'_1}, j'_{t'_1}; j''_{t''_1}, i''_{t''_1})B(j'_{t'_1}, i''_{t''_1}), \tag{3.21}$$

where $A(i'_{t'_1}, j''_{t''_1})$ equals

$$\sum_{i'_1,\dots,i'_{t'_1-1}} \sum_{j'_1,\dots,j'_{t'_1-1}} \left[\prod_{w=0}^{t'_1-1} a_w(j'_w, i'_{w+1})\right] \prod_{\substack{u\sim v\\ \pi'_0}} \sigma(i'_u, j'_u; i'_v, j'_v)\times$$

$$\times \sum_{i''_{t''_1+1},\dots,i''_n} \sum_{j''_{t''_1+1},\dots,j''_n} \left[\prod_{w=t''_1}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{\substack{u\sim v\\ \pi''_0+m+t''_1}} \sigma(i''_{u-m}, j''_{u-m}; i''_{v-m}, j''_{v-m}),$$

and $B(j'_{t'_1}, i''_{t''_1})$ equals

$$\sum_{\substack{i'_{t'_1+1},\dots,i'_m\\ j'_{t'_1+1},\dots,j'_m}} \sum_{\substack{i''_1,\dots,i''_{t''_1-1}\\ j''_1,\dots,j''_{t''_1-1}}} \left[\prod_{w=t'_1}^{m} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=0}^{t''_1-1} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{\substack{u\sim v\\ \tilde\pi}} \sigma(i_u, j_u; i_v, j_v),$$

with $j'_0 := p$, $i'_{m+1} := q$, $j''_0 := s$, and $i''_{n+1} := r$. Note that $u \sim v$ under $\pi''_0 + m + t''_1$ if and only if $u - m - t''_1 \sim v - m - t''_1$ under π''_0. The change of variables $u - m - t''_1 \to u$ and $v - m - t''_1 \to v$ leads to

$$\prod_{\substack{u\sim v\\ \pi''_0+m+t''_1}} \sigma(i''_{u-m}, j''_{u-m}; i''_{v-m}, j''_{v-m}) = \prod_{\substack{u\sim v\\ \pi''_0}} \sigma(i''_{u+t''_1}, j''_{u+t''_1}; i''_{v+t''_1}, j''_{v+t''_1}).$$

Similarly, the change of variable $w - t_1'' \to w$ leads to

$$\prod_{w=t_1''}^{n} b_{n-w}^{\mathrm{T}}(j_w'', i_{w+1}'') = \prod_{w=0}^{n-t_1''} b_{n-t_1''-w}^{\mathrm{T}}(j_{w+t_1''}'', i_{w+1+t_1''}'').$$

A straightforward change of variables and the base of induction then show that

$$A(i_{t_1'}', j_{t_1''}'') = \left[\kappa_{\pi_0'}(a_0, \ldots, a_{t_1'-1}) \otimes \kappa_{\widehat{\pi_0''}}(b_0, \ldots, b_{n-t_1''})\right](p, i_{t_1'}'; r, j_{t_1''}''). \tag{3.22}$$

Note that $\tilde{\pi} - t_1'$ is an element of $\mathrm{DP}_{m-t_1', t_1''-1}^{(k-1)}$. Mutatis mutandis, changing variables we obtain that

$$B(j_{t_1'}', i_{t_1''}'') = \kappa_{\tilde{\pi}-t_1'}(a_{t_1'}, \ldots, a_m; b_{n-t_1''+1}, \ldots, b_n)(j_{t_1'}', q; i_{t_1''}'', s). \tag{3.23}$$

Recall the product formula in (2.2). By plugging (3.22) and (3.23) in (3.21), we conclude that $\kappa_\pi(\mathbf{a}; \mathbf{b})$ equals

$$\left[\kappa_{\pi_0'}(a_0, \ldots, a_{t_1'-1}) \otimes \kappa_{\widehat{\pi_0''}}(b_0, \ldots, b_{n-t_1''})\right] \Sigma \kappa_{\tilde{\pi}-t_1'}(a_{t_1'}, \ldots, a_m; b_{n-t_1''+1}, \ldots, b_n).$$

The result follows after applying the induction hypothesis to $\kappa_{\tilde{\pi}-t_1'}$. □

With the usual abuse of notation, the previous lemma reads as

$$\kappa_{\Psi^{(k)}(\pi_0', \ldots, \pi_k'; \pi_k'', \ldots, \pi_0'')}(a; b) = \left[\kappa_{\pi_0'}(a) \otimes \kappa_{\widehat{\pi_0''}}(b)\right] \Sigma \cdots \Sigma \left[\kappa_{\pi_k'}(a) \otimes \kappa_{\widehat{\pi_k''}}(b)\right]. \tag{3.24}$$

As with SP, there is a Cauchy transform associated to DP.

Definition 12. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ.

We define the double-line Cauchy transform $G_D : \mathrm{M}_d \times \mathrm{M}_d \to \mathrm{M}_{d^2}$ of X by

$$G_D(a,b) = \sum_{\pi \in \mathrm{DP}} \kappa_\pi(a^{-1}; b^{-1}).$$

This Cauchy transform is closely related to the single-line Cauchy transform.

Proposition 7. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. The double-line Cauchy transform of X is given by

$$G_D(a,b) = [G_S(a) \otimes G_S(b)] \left(\mathrm{I}_{d^2} - \Sigma [G_S(a) \otimes G_S(b)]\right)^{-1}.$$

Proof. Observe that

$$G_D(a,b) = \sum_{\pi \in \mathrm{DP}} \kappa_\pi(a^{-1}; b^{-1}) = \sum_{k \geq 0} \sum_{\pi \in \mathrm{DP}^{(k)}} \kappa_\pi(a^{-1}; b^{-1}). \tag{3.25}$$

The bijection in Proposition 4 implies that

$$\sum_{\pi \in \mathrm{DP}^{(k)}} \kappa_\pi(a^{-1}; b^{-1}) = \sum_{\pi'_0, \ldots, \pi'_k \in \mathrm{SP}} \sum_{\pi''_0, \ldots, \pi''_k \in \mathrm{SP}} \kappa_{\Psi^{(k)}(\pi'_0, \ldots, \pi'_k, \pi''_k, \ldots, \pi''_0)}(a^{-1}; b^{-1}).$$

By equation (3.24), Definition 10 and the fact that $\{\hat{\pi} : \pi \in \mathrm{SP}\} = \mathrm{SP}$,

$$\begin{aligned}\sum_{\pi \in \mathrm{DP}^{(k)}} \kappa_\pi(a^{-1}; b^{-1}) &= \sum_{\substack{\pi'_0, \ldots, \pi'_k \in \mathrm{SP} \\ \pi''_0, \ldots, \pi''_k \in \mathrm{SP}}} \left[\kappa_{\pi'_0}(a^{-1}) \otimes \kappa_{\widehat{\pi''_0}}(b^{-1})\right] \Sigma \cdots \Sigma \left[\kappa_{\pi'_k}(a^{-1}) \otimes \kappa_{\widehat{\pi''_k}}(b^{-1})\right] \\ &= [G_S(a) \otimes G_S(b)] \left(\Sigma [G_S(a) \otimes G_S(b)]\right)^k.\end{aligned}$$

Plugging the previous equation in (3.25), we conclude that

$$G_D(a,b) = \sum_{k\geq 0} [G_S(a)\otimes G_S(b)] \left(\Sigma\left[G_S(a)\otimes G_S(b)\right]\right)^k$$
$$= [G_S(a)\otimes G_S(b)] \left(\mathrm{I}_{d^2} - \Sigma\left[G_S(a)\otimes G_S(b)\right]\right)^{-1},$$

as required. □

The following function will play an important role in the next section.

Definition 13. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. We define

$$H(a,b) = \sum_{\pi\in \mathrm{DP}^{||}} \kappa_\pi(\mathrm{I}_d, a^{-1}, \ldots, a^{-1}, \mathrm{I}_d; \mathrm{I}_d, b^{-1}, \ldots, b^{-1}, \mathrm{I}_d).$$

In this case, we have the following.

Proposition 8. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. Then

$$H(a,b) = (\mathrm{I}_{d^2} - \Sigma[G_S(a)\otimes G_S(b)])^{-1}\Sigma.$$

Proof. Recall the bijection between DP and $\mathrm{DP}^{||}\backslash \mathrm{DP}_{1,1}$ established at the end of Section 3.4.2, and denote it by $\mathrm{DP}_{m,n} \ni \pi \leftrightarrow \tilde{\pi} \in \mathrm{DP}_{m+2,n+2} \cap \mathrm{DP}^{||}$. Since $\kappa_\emptyset(a) = a$ for all $a \in \mathrm{M}_d$, equation (3.24) implies that

$$\kappa_{\tilde{\pi}}(\mathrm{I}_d, a^{-1}, \ldots, a^{-1}, \mathrm{I}_d; \mathrm{I}_d, b^{-1}, \ldots, b^{-1}, \mathrm{I}_d) = \Sigma\kappa_\pi(a^{-1}; b^{-1})\Sigma.$$

By the definition of $H(a,b)$, we obtain that

$$\begin{aligned} H(a,b) &= \Sigma + \sum_{\tilde{\pi}\in \mathrm{DP}^{||}\setminus \mathrm{DP}_{1,1}} \kappa_{\tilde{\pi}}(\mathrm{I}_d, a^{-1}, \ldots, a^{-1}, \mathrm{I}_d; \mathrm{I}_d, b^{-1}, \ldots, b^{-1}, \mathrm{I}_d) \\ &= \Sigma + \sum_{\pi\in \mathrm{DP}} \Sigma \kappa_\pi(a^{-1}; b^{-1})\Sigma \\ &= \Sigma + \Sigma G_D(a,b)\Sigma. \end{aligned}$$

Proposition 7 then leads to $H(a,b) = \left(\mathrm{I}_{d^2} - \Sigma\left[G_S(a)\otimes G_S(b)\right]\right)^{-1}\Sigma$. □

We finish this section with the following observation. Let $\pi \in \mathrm{DP}_{m,n}$. By Definition 11 we have that $\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s)$ equals

$$\sum_{\substack{i_1',\ldots,i_m' \\ j_1',\ldots,j_m'}} \sum_{\substack{i_1'',\ldots,i_n'' \\ j_1'',\ldots,j_n''}} \left[\prod_{w=0}^{m} a_w(j_w', i_{w+1}')\right] \left[\prod_{w=0}^{n} b_{n-w}^{\mathrm{T}}(j_w'', i_{w+1}'')\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v),$$

with $j_0' := p$, $i_{m+1}' := q$, $j_0'' := s$, and $i_{n+1}'' := r$. If $a_0 = a_m = b_0 = b_n = \mathrm{I}_d$, then $\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s)$ equals

$$\sum_{\substack{i_2',\ldots,i_m' \\ j_1',\ldots,j_{m-1}'}} \sum_{\substack{i_2'',\ldots,i_n'' \\ j_1'',\ldots,j_{n-1}''}} \left[\prod_{w=1}^{m-1} a_w(j_w', i_{w+1}')\right] \left[\prod_{w=1}^{n-1} b_{n-w}^{\mathrm{T}}(j_w'', i_{w+1}'')\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v),$$

where $i_1' := p$, $j_m' := q$, $i_1'' := s$, and $j_n'' := r$.

3.5.3 Annular Pairings

Motivated by (3.10), in this section we define the cumulants associated to the annular pairings and compute the corresponding Cauchy transform. Specifically, we have the following.

Definition 14. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. We define the cumulant $\kappa_\pi : \mathrm{M}_d^{m+1} \times \mathrm{M}_d^{n+1} \to \mathrm{M}_{d^2}$ associated to $\pi \in \mathrm{AP}_{m,n}$ by

$$\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s) = \sum_{i,j} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=0}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v),$$

where $j'_0 := p$, $i'_{m+1} := q$, $j''_0 := s$, and $i''_{n+1} := r$.

The name cumulant is justified by the fact that

$$P(a_0 X a_1 \cdots a_{m-1} X a_m \otimes b_0 X b_1 \cdots b_{n-1} X b_n) = \sum_{\pi \in \mathrm{AP}_{m,n}} \kappa_\pi(a_0, \ldots, a_m; b_0, \ldots, b_n)$$

whenever X is a MSO semicircular element with covariance σ, as established in Proposition 1. As we did before, we study AP via the classes $\mathcal{T}_{\mathrm{I}}$ and $\mathcal{T}_{\mathrm{II}}$.

Type I Annular Pairings

Let $\Theta : \mathrm{M}_{d^2} \to \mathrm{M}_{d^2}$ be the mapping determined by $\Theta(A)(p,q;r,s) = A(p,r;q,s)$. For example, when $d = 2$, then

$$\Theta \begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & D_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{21} & A_{22} \\ B_{11} & B_{12} & B_{21} & B_{22} \\ C_{11} & C_{12} & C_{21} & C_{22} \\ D_{11} & D_{12} & D_{21} & D_{22} \end{pmatrix}.$$

Lemma 4. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. If $\pi \in \mathrm{AP}^{(k)}_{m,n} \cap \mathcal{T}_{\mathrm{I}}$ and $\{t'_1, m + t''_1\}, \ldots, \{t'_k, m + t''_k\}$ are its through strings with

$t'_1 < \cdots < t'_k$ (so $t''_1 > \cdots > t''_k$), then $\kappa_\pi(\mathbf{a};\mathbf{b}) = \Theta\left(\kappa_{eb}\Theta[\kappa_{re}]\kappa_{ib}^{\mathrm{T}}\right)$ where

$$\begin{aligned}
\kappa_{re} &:= \kappa_{\pi_{re}}(\mathrm{I}_d, a_{t'_1}, \ldots, a_{t'_k-1}, \mathrm{I}_d; \mathrm{I}_d, b_{n-t''_1+1}, \ldots, b_{n-t''_k}, \mathrm{I}_d),\\
\kappa_{eb} &:= \kappa_{\pi_{eb}}(a_0, \ldots, a_{t'_1-1}; a_m^{\mathrm{T}}, \ldots, a_{t'_k}^{\mathrm{T}}),\\
\kappa_{ib} &:= \kappa_{\widehat{\pi_{ib}}}(b_0, \ldots, b_{n-t''_1}; b_n^{\mathrm{T}}, \ldots, b_{n-t''_k+1}^{\mathrm{T}}).
\end{aligned}$$

To illustrate the previous lemma, consider the following.

Example 5. Let π_1 be the Type I annular pairing in Figure 3.5. For $\mathbf{a} \in \mathrm{M}_d^{11}$ and $\mathbf{b} \in \mathrm{M}_d^9$,

$$\begin{aligned}
\kappa_{re} &= \Sigma[(a_2\eta(a_3\eta(a_4)a_5)a_6) \otimes (b_3\eta(b_4)b_5)]\Sigma,\\
\kappa_{eb} &= [a_0 \otimes a_{10}^{\mathrm{T}}]\Sigma[a_1 \otimes (a_9^{\mathrm{T}}\eta(a_8^{\mathrm{T}})a_7^{\mathrm{T}})],\\
\kappa_{ib} &= (b_0\eta(b_1)b_2) \otimes (b_8^{\mathrm{T}}\eta(b_7^{\mathrm{T}})b_6^{\mathrm{T}}).
\end{aligned}$$

We then obtain $\kappa_\pi(\mathbf{a},\mathbf{b})$ by plugging the previous equations in $\Theta\left(\kappa_{eb}\Theta[\kappa_{re}]\kappa_{ib}^{\mathrm{T}}\right)$.

Proof of Lemma 4. We decompose the sum in Definition 14 into three pieces. Fix $p, q, r, s \in [d]$. An appropriate change of variables, and the observation at the end of Section 3.5.2, shows that

$$\sum_{\substack{i'_{t'_1+1},\ldots,i'_{t'_k}\\ j'_{t'_1},\ldots,j'_{t'_k-1}}} \sum_{\substack{i''_{t''_k+1},\ldots,i''_{t''_1}\\ j''_{t''_k},\ldots,j''_{t''_1-1}}} \left[\prod_{w=t'_1}^{t'_k-1} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=t''_k}^{t''_1-1} b_{n-w}^{\mathrm{T}}(j''_w, i''_{w+1})\right] \prod_{u \underset{\pi_{re}}{\sim} v} \sigma(i_u, j_u; i_v, j_v)$$

equals $\kappa_{\pi_{re}}(\mathrm{I}_d, a_{t'_1}, \ldots, a_{t'_k-1}, \mathrm{I}_d; \mathrm{I}_d, b_{n-t''_1+1}, \ldots, b_{n-t''_k}, \mathrm{I}_d)(i'_{t'_1}, j'_{t'_k}; j''_{t''_1}, i''_{t''_k})$, which in turn equals $\kappa_{re}(i'_{t'_1}, j'_{t'_k}; j''_{t''_1}, i''_{t''_k})$. As usual, let $j'_0 := p$ and $i'_{n+1} := q$. Observe that

$$\sum_{\substack{i'_1,\ldots,i'_{t'_1-1}\\ j'_1,\ldots,j'_{t'_1-1}}} \sum_{\substack{i'_{t'_k+1},\ldots,i'_n\\ j'_{t'_k+1},\ldots,j'_n}} \left[\prod_{w=0}^{t'_1-1} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=t'_k}^{m} a_w(j'_w, i'_{w+1})\right] \prod_{\substack{u\sim v\\ \pi_{eb}}} \sigma(i_u, j_u; i_v, j_v) \quad (3.26)$$

can be written as

$$\sum_{\substack{i'_1,\ldots,i'_{t'_1-1}\\ j'_1,\ldots,j'_{t'_1-1}}} \sum_{\substack{i'_{t'_k+1},\ldots,i'_n\\ j'_{t'_k+1},\ldots,j'_n}} \left[\prod_{w=0}^{t'_1-1} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=t'_k}^{m} c^{\mathrm{T}}_{m-w}(j'_w, i'_{w+1})\right] \prod_{\substack{u\sim v\\ \pi_{eb}}} \sigma(i_u, j_u; i_v, j_v),$$

where $c_{m-w} = a^{\mathrm{T}}_w$ for $t'_k \le w \le m$. After a suitable change of variables, Definition 11 shows that (3.26) equals $\kappa_{\pi_{eb}}(a_0, \ldots, a_{t'_1-1}; c_0, \ldots, c_{m-t'_k})(p, i'_{t'_1}; q, j'_{t'_k})$, which equals

$$\kappa_{\pi_{eb}}(a_0, \ldots, a_{t'_1-1}; a^{\mathrm{T}}_m, \ldots, a^{\mathrm{T}}_{t'_k})(p, i'_{t'_1}; q, j'_{t'_k}) = \kappa_{eb}(p, i'_{t'_1}; q, j'_{t'_k}).$$

Let $j''_0 := s$ and $i''_{n+1} := r$. Similarly as with π_{eb}, we have that

$$\sum_{\substack{i''_1,\ldots,i''_{t''_k-1}\\ j''_1,\ldots,j''_{t''_k-1}}} \sum_{\substack{i''_{t''_1+1},\ldots,i''_n\\ j''_{t''_1+1},\ldots,j''_n}} \left[\prod_{w=0}^{t''_k-1} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \left[\prod_{k=t''_1}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{\substack{u\sim v\\ \pi_{ib}}} \sigma(i_u, j_u; i_v, j_v)$$

equals $\kappa_{\pi_{ib}}(b^{\mathrm{T}}_n, \ldots, b^{\mathrm{T}}_{n-t''_k+1}; b_0, \ldots, b_{n-t''_1})(s, i''_{t''_k}; r, j''_{t''_1})$. By Lemma 2, it equals

$$\kappa_{\widehat{\pi_{ib}}}(b_0, \ldots, b_{n-t''_1}; b^{\mathrm{T}}_n, \ldots, b^{\mathrm{T}}_{n-t''_k+1})(r, j''_{t''_1}; s, i''_{t''_k}) = \kappa_{ib}(r, j''_{t''_1}; s, i''_{t''_k}).$$

Altogether, we obtain that

$$\begin{aligned}
\kappa_\pi(\mathbf{a};\mathbf{b})(p,q;r,s) &= \sum_{i'_{t'_1},j'_{t'_k},i''_{t''_k},j''_{t''_1}} \kappa_{eb}(p,i'_{t'_1};q,j'_{t'_k})\kappa_{re}(i'_{t'_1},j'_{t'_k};j''_{t''_1},i''_{t''_k})\kappa_{ib}(r,j''_{t''_1};s,i''_{t''_k}) \\
&= \sum_{i'_{t'_1},j'_{t'_k},i''_{t''_k},j''_{t''_1}} \kappa_{eb}(p,i'_{t'_1};q,j'_{t'_k})\Theta[\kappa_{re}](i'_{t'_1},j''_{t''_1};j'_{t'_k},i''_{t''_k})\kappa_{ib}^{\mathrm{T}}(j''_{t''_1},r;i''_{t''_k},s) \\
&= [\kappa_{eb}\Theta[\kappa_{re}]\kappa_{ib}^{\mathrm{T}}](p,r;q,s) \\
&= \Theta(\kappa_{eb}\Theta[\kappa_{re}]\kappa_{ib}^{\mathrm{T}})(p,q;r,s).
\end{aligned}$$

Since $p,q,r,s \in [d]$ are arbitrary, we conclude that $\kappa_\pi(\mathbf{a};\mathbf{b}) = \Theta(\kappa_{eb}\Theta[\kappa_{re}]\kappa_{ib}^{\mathrm{T}})$. □

In particular, for $a,b \in \mathrm{M}_d$,

$$\kappa_\pi(a;b) = \Theta\left(\kappa_{\pi_{eb}}(a;a^{\mathrm{T}})\Theta[\kappa_{\pi_{re}}(\mathrm{I}_d,a,\ldots,a,\mathrm{I}_d;\mathrm{I}_d,b,\ldots,b,\mathrm{I}_d)]\kappa_{\widehat{\pi_{ib}}}(b;b^{\mathrm{T}})^{\mathrm{T}}\right). \quad (3.27)$$

As before, we get a rather neat expression when we sum over all π in $\mathcal{T}_{\mathrm{I}}$.

Proposition 9. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. Then

$$\sum_{\pi\in\mathcal{T}_{\mathrm{I}}} \kappa_\pi(a^{-1};b^{-1}) = \Theta\left(G_D(a,a^{\mathrm{T}})\Theta[H(a,b)]G_D(b,b^{\mathrm{T}})^{\mathrm{T}}\right).$$

Proof. By Proposition 5, the map $\mathcal{T}_{\mathrm{I}} \ni \pi \mapsto (\pi_{eb},\pi_{ib},\pi_{re}) \in \mathrm{DP}\times\mathrm{DP}\times\mathrm{DP}^{||}$ is a bijection. In particular,

$$\sum_{\pi\in\mathcal{T}_{\mathrm{I}}} \kappa_\pi(a^{-1};b^{-1}) = \sum_{\pi_{eb}\in\mathrm{DP}}\sum_{\pi_{ib}\in\mathrm{DP}}\sum_{\pi_{re}\in\mathrm{DP}^{||}} \kappa_{(\pi_{eb},\pi_{ib},\pi_{re})}(a^{-1};b^{-1}).$$

By equation (3.27), we have that $\sum_{\pi \in \mathcal{T}_{\mathrm{I}}} \kappa_\pi(a^{-1}; b^{-1})$ equals

$$\sum_{\substack{\pi_{eb} \in \mathrm{DP} \\ \pi_{ib} \in \mathrm{DP} \\ \pi_{re} \in \mathrm{DP}^{||}}} \Theta\left(\kappa_{\pi_{eb}}(a^{-1}; a^{-\mathrm{T}}) \Theta[\kappa_{\pi_{re}}(\mathrm{I}_d, a^{-1}, \ldots, a^{-1}, \mathrm{I}_d; \mathrm{I}_d, b^{-1}, \ldots, b^{-1}, \mathrm{I}_d)] \kappa_{\widehat{\pi_{ib}}}(b^{-1}; b^{-\mathrm{T}})^{\mathrm{T}}\right),$$

where $a^{-\mathrm{T}} = (a^{\mathrm{T}})^{-1}$. Since Θ is linear and $\{\hat{\pi} : \pi \in \mathrm{DP}\} = \mathrm{DP}$, we obtain that

$$\sum_{\pi \in \mathcal{T}_{\mathrm{I}}} \kappa_\pi(a^{-1}; b^{-1}) = \Theta\left(G_D(a, a^{\mathrm{T}}) \Theta[H(a, b)] G_D(b, b^{\mathrm{T}})^{\mathrm{T}}\right),$$

as required. $\square$

Type II Annular Pairings

Let $A^\Gamma(p, q; r, s) = A(p, q; s, r)$ for all $A \in \mathrm{M}_{d^2}$ and $p, q, r, s \in [d]$. Also, let $\Phi(A) = \Theta(A^\Gamma)$.

Lemma 5. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. If $\pi \in \mathrm{AP}^{(k)}_{m,n} \cap \mathcal{T}_{\mathrm{II}}$ and $\{t'_1, m + t''_1\}, \ldots, \{t'_k, m + t''_k\}$ are its through strings with $t'_1 < \cdots < t'_k$ and $t''_{s+1} > \cdots > t''_k > t''_1 > \cdots > t''_s$ for some $s \in [k]$, then

$$\kappa_\pi(\mathbf{a}; \mathbf{b}) = \Theta(\kappa_{eb} \Phi[\kappa^\Gamma_{rr}(\kappa_{oi} \otimes \kappa_{oe})^\Gamma \kappa^\Gamma_{lr}] \kappa^{\mathrm{T}}_{ib}),$$

where

$$\kappa_{oi} := \kappa_{\pi_{oi}}(a_{t'_s}, \ldots, a_{t'_{s+1}-1}),$$

$$\kappa_{oe} := \kappa_{\widehat{\pi_{oe}}}(b_{n-t''_k+1}, \ldots, b_{n-t''_1}),$$

$$\kappa_{rr} := \kappa_{\pi_{rr}}(\mathrm{I}_d, a_{t'_1}, \ldots, a_{t'_s-1}, \mathrm{I}_d; \mathrm{I}_d, b_{n-t''_1+1}, \ldots, b_{n-t''_s}, \mathrm{I}_d),$$

$$\kappa_{lr} := \kappa_{\pi_{rl}}(\mathrm{I}_d, a_{t'_{s+1}}, \ldots, a_{t'_k-1}, \mathrm{I}_d; \mathrm{I}_d, b_{n-t''_{s+1}+1}, \ldots, b_{n-t''_k}, \mathrm{I}_d),$$

$$\kappa_{eb} := \kappa_{\pi_{eb}}(a_0, \ldots, a_{t'_1-1}; a_m^{\mathrm{T}}, \ldots, a_{t'_k}^{\mathrm{T}}),$$

$$\kappa_{ib} = \kappa_{\widehat{\pi_{ib}}}(b_0, \ldots, b_{n-t''_{s+1}}; b_n^{\mathrm{T}}, \ldots, b_{n-t''_s+1}^{\mathrm{T}}).$$

In this case, $\kappa_\pi(\mathbf{a}, \mathbf{b})$ can be constructed, mutatis mutandis, as in Example 5. In fact, the next proof is very similar to that of Lemma 4.

Proof. A change of variable shows that

$$\sum_{i'_{t'_s+1},\ldots,i'_{t'_{s+1}-1}} \sum_{j'_{t'_s+1},\ldots,j'_{t'_{s+1}-1}} \left[\prod_{w=t'_s}^{t'_{s+1}-1} a_w(j'_w, i'_{w+1})\right] \prod_{\substack{u\sim v\\ \pi_{oi}}} \sigma(i_u, j_u; i_v, j_v)$$

equals

$$\kappa^\sigma_{\pi_{oi}}(a_{t'_s}, \ldots, a_{t'_{s+1}-1})(j'_{t'_s}, i'_{t'_{s+1}}) = \kappa_{oi}(j'_{t'_s}, i'_{t'_{s+1}}).$$

Recall the property of the single-line cumulants established in Lemma 1. As with π_{oi}, we have that

$$\sum_{i''_{t''_1+1},\ldots,i''_{t''_k-1}} \sum_{j''_{t''_1+1},\ldots,j''_{t''_k-1}} \left[\prod_{w=t''_1}^{t''_k-1} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{\substack{u\sim v\\ \pi_{oe}}} \sigma(i_u, j_u; i_v, j_v)$$

equals

$$\kappa_{\pi_{oe}}(b^{\mathrm{T}}_{n-t''_1}, \ldots, b^{\mathrm{T}}_{n-t''_k+1})(j''_{t''_1}, i''_{t''_k}) = \kappa_{\widehat{\pi_{oe}}}(b_{n-t''_k+1}, \ldots, b_{n-t''_1})(i''_{t''_k}, j''_{t''_1}) = \kappa^{\mathrm{T}}_{oe}(j''_{t''_1}, i''_{t''_k}).$$

As in Lemma 4, we have that

$$\sum_{\substack{i'_{t'_1+1},\dots,i'_{t'_s}\\ j'_{t'_1},\dots,j'_{t'_s-1}}} \sum_{\substack{i''_{t''_s+1},\dots,i''_{t''_1}\\ j''_{t''_s},\dots,j''_{t''_1-1}}} \left[\prod_{w=t'_1}^{t'_s-1} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=t''_s}^{t''_1-1} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{\substack{u \sim v\\ \pi_{rr}}} \sigma(i_u, j_u; i_v, j_v)$$

equals

$$\kappa_{\pi_{rr}}(\mathrm{I}_d, a_{t'_1}, \dots, a_{t'_s-1}, \mathrm{I}_d; \mathrm{I}_d, b_{n-t''_1+1}, \dots, b_{n-t''_s}, \mathrm{I}_d)(i'_{t'_1}, j'_{t'_s}; j''_{t''_1}, i''_{t''_s}) = \kappa_{rr}(i'_{t'_1}, j'_{t'_s}; j''_{t''_1}, i''_{t''_s}),$$

and

$$\sum_{\substack{i'_{t'_{s+1}+1},\dots,i'_{t'_k}\\ j'_{t'_{s+1}},\dots,j'_{t'_k-1}}} \sum_{\substack{i''_{t''_k+1},\dots,i''_{t''_{s+1}}\\ j''_{t''_k},\dots,j''_{t''_{s+1}-1}}} \left[\prod_{w=t'_{s+1}}^{t'_k-1} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=t''_k}^{t''_{s+1}-1} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{\substack{u \sim v\\ \pi_{lr}}} \sigma(i_u, j_u; i_v, j_v)$$

equals

$$\kappa_{\pi_{lr}}(\mathrm{I}_d, a_{t'_{s+1}}, \dots, a_{t'_k-1}, \mathrm{I}_d; \mathrm{I}_d, b_{n-t''_{s+1}+1}, \dots, b_{n-t''_k}, \mathrm{I}_d)(i'_{t'_{s+1}}, j'_{t'_k}; j''_{t''_{s+1}}, i''_{t''_k}).$$

Note that, by definition, the last expression equals $\kappa_{lr}(i'_{t'_{s+1}}, j'_{t'_k}; j''_{t''_{s+1}}, i''_{t''_k})$. In an analogous way, we have that

$$\sum_{\substack{i'_1,\dots,i'_{t'_1-1}\\ j'_1,\dots,j'_{t'_1-1}}} \sum_{\substack{i'_{t'_k+1},\dots,i'_n\\ j'_{t'_k+1},\dots,j'_n}} \left[\prod_{w=0}^{t'_1-1} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=t'_k}^{m} a_w(j'_w, i'_{w+1})\right] \prod_{\substack{u \sim v\\ \pi_{eb}}} \sigma(i_u, j_u; i_v, j_v)$$

equals $\kappa_{\pi_{eb}}(a_0, \ldots, a_{t'_1-1}; a_m^{\mathrm{T}}, \ldots, a_{t'_k}^{\mathrm{T}})(p, i'_{t'_1}; q, j'_{t'_k}) = \kappa_{eb}(p, i'_{t'_1}; q, j'_{t'_k})$, and

$$\sum_{\substack{i''_1,\ldots,i''_{t''_s-1}\\ j''_1,\ldots,j''_{t''_s-1}}} \sum_{\substack{i''_{t''_{s+1}+1},\ldots,i''_n\\ j''_{t''_{s+1}+1},\ldots,j''_n}} \left[\prod_{w=0}^{t''_s-1} b_{n-w}^{\mathrm{T}}(j''_w, i''_{w+1})\right] \left[\prod_{k=t''_{s+1}}^{n} b_{n-w}^{\mathrm{T}}(j''_w, i''_{w+1})\right] \prod_{\substack{u\sim v\\ \pi_{ib}}} \sigma(i_u, j_u; i_v, j_v)$$

equals

$$\kappa_{\widehat{\pi_{ib}}}(b_0, \ldots, b_{n-t''_{s+1}}; b_n^{\mathrm{T}}, \ldots, b_{n-t''_s+1}^{\mathrm{T}})(r, j''_{t''_{s+1}}; s, i''_{t''_s}) = \kappa_{ib}(r, j''_{t''_{s+1}}; s, i''_{t''_s}).$$

Altogether, $\kappa_\pi(\mathbf{a}; \mathbf{b})(p, q; r, s)$ equals

$$\begin{aligned}
&\sum_{i'_{t'_1}, j'_{t'_k}, i''_{t''_s}, j''_{t''_{s+1}}} \kappa_{eb}(p, i'_{t'_1}; q, j'_{t'_k}) \kappa_{ib}(r, j''_{t''_{s+1}}; s, i''_{t''_s}) \times \\
&\qquad \times \sum_{j'_{t'_s}, i'_{t'_{s+1}}, j''_{t''_1}, i''_{t''_k}} \kappa_{rr}(i'_{t'_1}, j'_{t'_s}; j''_{t''_1}, i''_{t''_s}) \kappa_{oi}(j'_{t'_s}, i'_{t'_{s+1}}) \kappa_{oe}^{\mathrm{T}}(j''_{t''_1}, i''_{t''_k}) \kappa_{lr}(i'_{t'_{s+1}}, j'_{t'_k}; j''_{t''_{s+1}}, i''_{t''_k}),
\end{aligned}$$

which in turn equals

$$\begin{aligned}
&\sum_{i'_{t'_1}, j'_{t'_k}, i''_{t''_s}, j''_{t''_{s+1}}} \kappa_{eb}(p, i'_{t'_1}; q, j'_{t'_k}) [\kappa_{rr}^{\Gamma}(\kappa_{oi} \otimes \kappa_{oe}^{\mathrm{T}}) \kappa_{lr}^{\Gamma}](i'_{t'_1}, j'_{t'_k}; i''_{t''_s}, j''_{t''_{s+1}}) \kappa_{ib}^{\mathrm{T}}(j''_{t''_{s+1}}, r; i''_{t''_s}, s) \\
&\quad = \sum_{i'_{t'_1}, j'_{t'_k}, i''_{t''_s}, j''_{t''_{s+1}}} \kappa_{eb}(p, i'_{t'_1}; q, j'_{t'_k}) \Phi[\kappa_{rr}^{\Gamma}(\kappa_{oi} \otimes \kappa_{oe}^{\mathrm{T}}) \kappa_{lr}^{\Gamma}](i'_{t'_1}, j''_{t''_{s+1}}; j'_{t'_k}, i''_{t''_s}) \kappa_{ib}^{\mathrm{T}}(j''_{t''_{s+1}}, r; i''_{t''_s}, s) \\
&\quad = \kappa_{eb} \Phi[\kappa_{rr}^{\Gamma}(\kappa_{oi} \otimes \kappa_{oe}^{\mathrm{T}}) \kappa_{lr}^{\Gamma}] \kappa_{ib}^{\mathrm{T}}(p, r; q, s) \\
&\quad = \Theta(\kappa_{eb} \Phi[\kappa_{rr}^{\Gamma}(\kappa_{oi} \otimes \kappa_{oe}^{\mathrm{T}}) \kappa_{lr}^{\Gamma}] \kappa_{ib}^{\mathrm{T}})(p, q; r, s).
\end{aligned}$$

Since $p, q, r, s \in [d]$ are arbitrary, we conclude that

$$\kappa_\pi(\mathbf{a}; \mathbf{b}) = \Theta(\kappa_{eb}\Phi[\kappa_{rr}^\Gamma(\kappa_{oi} \otimes \kappa_{oe})^\Gamma \kappa_{lr}^\Gamma]\kappa_{ib}^{\mathrm{T}}),$$

as required. ☐

In particular, $\Theta\left(\kappa_\pi(a; b)\right)$ equals

$$\kappa_{\pi_{eb}}(a; a^{\mathrm{T}})\Phi[\kappa_{\pi_{rr}}(\mathrm{I}_d, a, \ldots, \mathrm{I}_d; \mathrm{I}_d, b, \ldots, \mathrm{I}_d)^\Gamma(\kappa_{\pi_{oi}}(a) \otimes \kappa_{\widehat{\pi_{oe}}}(b))^\Gamma \times \\ \times \kappa_{\pi_{lr}}(\mathrm{I}_d, a, \ldots, \mathrm{I}_d; \mathrm{I}_d, b, \ldots, \mathrm{I}_d)^\Gamma]\kappa_{\widehat{\pi_{ib}}}(b; b^{\mathrm{T}})^{\mathrm{T}}.$$

As before, we get a rather neat expression when we sum over all π in $\mathcal{T}_{\mathrm{II}}$.

Proposition 10. Let $X \in \mathrm{M}_d\left(\mathcal{A}\right)$ be a MSO semicircular element with covariance σ. Then

$$\sum_{\pi \in \mathcal{T}_{\mathrm{II}}} \kappa_\pi(a^{-1}; b^{-1}) = \Theta\left(G_D(a, a^{\mathrm{T}})\Phi\left[H(a, b)^\Gamma(G_S(a) \otimes G_S(b))^\Gamma H(a, b)^\Gamma\right] G_D(b, b^{\mathrm{T}})^{\mathrm{T}}\right).$$

Proof. Observe that the mappings $\Phi(\cdot)$ and $(\cdot)^\Gamma$ are both linear. Using the bijection established in Proposition 6, the previous lemma implies the result as in the proof of Proposition 9. ☐

As before, consider the following.

Definition 15. Let $X \in \mathrm{M}_d\left(\mathcal{A}\right)$ be a MSO semicircular element with covariance σ. We define the annular Cauchy transform $G_A : \mathrm{M}_d \times \mathrm{M}_d \to \mathrm{M}_{d^2}$ of X by

$$G_A(a, b) = \sum_{\pi \in \mathrm{AP}} \kappa_\pi(a^{-1}; b^{-1}).$$

Our main combinatorial theorem is now a simple consequence Propositions 9 and 10.

Theorem 3. Let $X \in \mathrm{M}_d(\mathcal{A})$ be a MSO semicircular element with covariance σ. Then $G_A(a,b)$ equals

$$\Theta\Bigg(G_D(a,a^{\mathrm{T}})\Bigg\{\Theta[H(a,b)] + \Phi\left[H(a,b)^{\Gamma}(G_S(a)\otimes G_S(b))^{\Gamma}H(a,b)^{\Gamma}\right]\Bigg\}G_D(b,b^{\mathrm{T}})^{\mathrm{T}}\Bigg).$$

All the results in this section were derived at the level of formal expressions, i.e., in the algebra of $d^2 \times d^2$ matrices over the formal power series in $2d^2$ commuting variables. The purpose of the next section is to extend these results to an analytical level.

3.6 Some Analytical Properties of the MSO Cauchy Transform of MSO Semicircular Elements

Assume that X is a MSO semicircular element. By Corollary 1, its MSO Cauchy transform $\mathcal{G}_2$ equals the annular Cauchy transform G_A. In particular, Theorem 3 implies that $\mathcal{G}_2(a,b)$ equals

$$\Theta\Bigg(G_D(a,a^{\mathrm{T}})\Bigg\{\Theta[H(a,b)] + \Phi\left[H(a,b)^{\Gamma}(G_S(a)\otimes G_S(b))^{\Gamma}H(a,b)^{\Gamma}\right]\Bigg\}G_D(b,b^{\mathrm{T}})^{\mathrm{T}}\Bigg). \tag{3.28}$$

In this section we explore the analytical properties of the right hand side of the previous equation. In order to do so, we need to assume an underlying analytic framework.

We assume that our second-order probability space $(\mathcal{A},\varphi,\rho)$ satisfies that

$$\mathcal{A} = \mathbb{C}\langle X(p,q) : p,q \in [d]\rangle$$

and that $X = (X(p,q))_{p,q} \in \mathrm{M}_d(\mathcal{A})$ is a MSO semicircular element. Also, we assume that there exists a tracial C*-probability space $(\mathcal{B}, \psi)$ such that $\mathcal{A} \subset \mathcal{B}$ and $\varphi = \psi|_{\mathcal{A}}$. Note that even though φ can be extended to the whole C*-algebra $\mathcal{B}$, the bilinear functional ρ is only defined on $\mathcal{A} \times \mathcal{A}$. In fact, in some classical examples [13] the bilinear functional ρ cannot be extended continuously to the C*-algebra generated by $\mathcal{A}$. We further assume that X and M_d are algebraically free, so that monomials $a_0 X \cdots X a_n$ with $a_0, \ldots, a_n \in \mathrm{M}_d$ are linearly independent from each other whenever they have different lengths n. By definition, the domain of P contains $\mathrm{M}_d\langle X\rangle \times \mathrm{M}_d\langle X\rangle$. Observe that the system

$$\mathrm{M}_d\langle X\rangle := \mathrm{Span}_{\mathbb{C}}\{a_0 X \cdots X a_n \colon n \in \mathbb{N}, a_j \in \mathrm{M}_d\}$$

is norm-dense in the C*-algebra generated by X and M_d. Similarly as with ρ, the mapping P may be unbounded in the norm topology of the C*-algebra generated by X and M_d. Establishing analytical properties of $\mathcal{G}_2$ accounts to extending P beyond $\mathrm{M}_d\langle X\rangle \times \mathrm{M}_d\langle X\rangle$. The following lemma establishes the bounded behavior of P in monomials from $\mathrm{M}_d\langle X\rangle$.

Lemma 6. If $X \in \mathrm{M}_d(\mathcal{A})$ is a MSO semicircular element with covariance σ, then

$$\|P(a_0 X \cdots X a_m \otimes b_0 X \cdots X b_n)\|$$

is bounded from above by

$$d\left(2d^2\sqrt{\|\Sigma\|}\right)^m \left(2d^2\sqrt{\|\Sigma\|}\right)^n \|a_0\| \cdots \|a_m\| \cdot \|b_0\| \cdots \|b_n\|$$

for all $m, n \in \mathbb{N}$ and $a_0, \dots, a_m, b_0, \dots, b_n \in \mathrm{M}_d$.

Proof. Recall that $M_{m,n}(\mathbf{a};\mathbf{b}) = P(a_0 X \cdots X a_m \otimes b_0 X \cdots X b_n)$. Fix $p, q, r, s \in [d]$. From Proposition 1, we have that $M_{m,n}(\mathbf{a};\mathbf{b})(p,q;r,s)$ equals

$$\sum_{\pi \in \mathrm{AP}_{m,n}} \sum_{\substack{i'_1,\dots,i'_m \\ j'_1,\dots,j'_m}} \sum_{\substack{i''_1,\dots,i''_n \\ j''_1,\dots,j''_n}} \left[\prod_{w=0}^{m} a_w(j'_w, i'_{w+1})\right] \left[\prod_{w=0}^{n} b^{\mathrm{T}}_{n-w}(j''_w, i''_{w+1})\right] \prod_{u \underset{\pi}{\sim} v} \sigma(i_u, j_u; i_v, j_v)$$

with $j'_0 = p$, $i'_{m+1} = q$, $j''_0 = s$, and $i''_{n+1} = r$. Since $|A(i,j)| \le \|A\|$ for all $A \in \mathrm{M}_d$ and all $1 \le i, j \le d$,

$$|M_{m,n}(\mathbf{a};\mathbf{b})(p,q;r,s)| \le \left[\prod_{w=0}^{m} \|a_w\|\right] \left[\prod_{w=0}^{n} \|b_w\|\right] \|\Sigma\|^{(m+n)/2} d^{2m} d^{2n} |\mathrm{AP}_{m,n}|. \quad (3.29)$$

Recall that [37, eq. (11)], for every $m, n \ge 1$,

$$|\mathrm{AP}_{m,n}| = \begin{cases} \frac{mn}{2(m+n)} \binom{m}{m/2} \binom{n}{n/2} & m, n \text{ even}, \\ \frac{(m+1)(n+1)}{8(m+n)} \binom{m+1}{(m+1)/2} \binom{n+1}{(n+1)/2} & m, n \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

The standard estimate $\binom{2n}{n} \le 2^{2n}/\sqrt{2n}$ (e.g., [45]) readily implies that $|\mathrm{AP}_{m,n}| \le 2^m 2^n$. Plugging this inequality in (3.29), we obtain that

$$|M_{m,n}(\mathbf{a};\mathbf{b})(p,q;r,s)| \le \left(2d^2\sqrt{\|\Sigma\|}\right)^m \left(2d^2\sqrt{\|\Sigma\|}\right)^n \left[\prod_{w=0}^{m} \|a_w\|\right] \left[\prod_{w=0}^{n} \|b_w\|\right].$$

In particular, $\|M_{m,n}(\mathbf{a};\mathbf{b})\|_{\max}$ is bounded by the same quantity and therefore

$$\|M_{m,n}(\mathbf{a};\mathbf{b})\| \leq d\left(2d^2\sqrt{\|\Sigma\|}\right)^m\left(2d^2\sqrt{\|\Sigma\|}\right)^n\|a_0\|\cdots\|a_m\|\cdot\|b_0\|\cdots\|b_n\|,$$

as we wanted to prove. □

With the usual abuse of notation, the previous lemma reads as

$$\|M_{m,n}(a;b)\| \leq d\left(2d^2\sqrt{\|\Sigma\|}\right)^m\left(2d^2\sqrt{\|\Sigma\|}\right)^n\|a\|^{m+1}\|b\|^{n+1}.$$

If a and b are invertible and $\|a^{-1}\|, \|b^{-1}\| < \left(2d^2\sqrt{\|\Sigma\|}\right)^{-1}$, then the series

$$\mathcal{G}_2(a,b) = \sum_{m,n=0}^{\infty} P\left((a^{-1}X)^n a^{-1}\otimes(b^{-1}X)^m b^{-1}\right) \tag{3.30}$$

converges in the C*-norm of $\mathrm{M}_d\otimes\mathrm{M}_d$. Thus, even though $\mathcal{G}_2(a,b)$ was defined at the level of formal expressions, it has a well defined meaning as a matricial power series for such a and b. By our standing assumptions, the mapping P is only defined on $\mathrm{M}_d(\mathcal{A})\times\mathrm{M}_d(\mathcal{A})$. Nonetheless, (3.30) allows us to define

$$P\left((a-X)^{-1}\otimes(b-X)^{-1}\right) := \sum_{m,n=0}^{\infty} P\left((a^{-1}X)^n a^{-1}, (b^{-1}X)^m b^{-1}\right) = \mathcal{G}_2(a,b)$$

whenever $a, b, a-X, b-X$ are invertible in $\mathrm{M}_d(\mathcal{B})$ and $\|a^{-1}\|, \|b^{-1}\| < \left(2d^2\sqrt{\|\Sigma\|}\right)^{-1}$.

In order to lift the equality in (3.28) to an analytic level, we introduce the following transforms (see p. 21 for the notation E):

$$\mathcal{G}(a) \quad = \quad E\left((a-X)^{-1}\right) = (\mathbf{1}\otimes\varphi)\left((a-X)^{-1}\right),$$

$$
\begin{aligned}
\mathcal{G}_D(a,b) &= [\mathcal{G}(a)\otimes\mathcal{G}(b)]\left(\mathrm{I}_{d^2}-\Sigma[\mathcal{G}(a)\otimes\mathcal{G}(b)]\right)^{-1},\\
\mathcal{H}(a,b) &= \left(\mathrm{I}_{d^2}-\Sigma[\mathcal{G}(a)\otimes\mathcal{G}(b)]\right)^{-1}\Sigma.
\end{aligned}
$$

Since $\mathcal{G}(a^{-1}) = E\left((a^{-1}-X)^{-1}\right) = E\left((1-aX)^{-1}\right)a$, we conclude that $a \mapsto \mathcal{G}(a^{-1})$ extends analytically to the ball $\{a \in \mathrm{M}_d\colon \|a\| < \|X\|^{-1}\}$. We will agree to denote this extension by $\mathcal{G}(a^{-1})$ even when a is not invertible. Since $\mathcal{G}(a^{-1}) \simeq a$ in norm near zero, there exists a neighborhood of zero, say B_0, such that the mappings $(a,b) \mapsto \mathcal{G}_D(a^{-1},b^{-1})$ and $(a,b) \mapsto \mathcal{H}(a^{-1},b^{-1})$ extend analytically to $B_0 \times B_0$. Note that if $a,b \in \mathrm{M}_d$ are invertible and $a^{-1},b^{-1} \in B_0$, then $G_S(a)$, $G_D(a,b)$ and $H(a,b)$ are convergent power series with limits $\mathcal{G}(a)$, $\mathcal{G}_D(a,b)$ and $\mathcal{H}(a,b)$, respectively. Therefore, equation (3.28) implies that for such a and b we have that $\Theta P((a-X)^{-1}\otimes(b-X)^{-1})$ equals

$$
\mathcal{G}_D(a,a^{\mathrm{T}})\left\{\Theta[\mathcal{H}(a,b)] + \Phi\left[\mathcal{H}(a,b)^{\Gamma}(\mathcal{G}(a)\otimes\mathcal{G}(b))^{\Gamma}\mathcal{H}(a,b)^{\Gamma}\right]\right\}\mathcal{G}_D(b,b^{\mathrm{T}})^{\mathrm{T}}
$$

in the sense of analytic mappings on $\mathrm{M}_d \times \mathrm{M}_d$. This equality allows to extend the domain of definition of P to be equal to the domain of the right hand side. In this direction, the main result of this section is the following. Recall that $a^{-\mathrm{T}} = (a^{\mathrm{T}})^{-1}$.

Theorem 4. For $a,b \in \mathrm{M}_d$ invertible, let

$$
\mathcal{D} = \{(z,w) \in \mathbb{C}^2\colon 1 - zaX, 1 - wbX \text{ invertible}\}.
$$

The mapping

$$
(z,w) \mapsto \mathcal{G}_D((za)^{-1},(za)^{-\mathrm{T}})\Big\{\Theta[\mathcal{H}((za)^{-1},(wb)^{-1})] +
$$

$$+ \Phi\left[\mathcal{H}((za)^{-1},(wb)^{-1})^{\Gamma}(\mathcal{G}((za)^{-1})\otimes\mathcal{G}((wb)^{-1}))^{\Gamma}\mathcal{H}((za)^{-1},(wb)^{-1})^{\Gamma}\right]\Big\}\times$$

$$\times\,\mathcal{G}_D((wb)^{-1},(wb)^{-\mathrm{T}})^{\mathrm{T}}$$

is analytic on the connected component of $\mathcal{D}$ containing $(0,0)$.

This theorem immediately leads to the following. By abuse of notation, we let z denote $z\mathrm{I}_d$.

Corollary 2. The map

$$(z,w)\mapsto \mathcal{G}_D(z,z)\left\{\Theta[\mathcal{H}(z,w)]+\Phi\left[\mathcal{H}(z,w)^{\Gamma}(\mathcal{G}(z)\otimes\mathcal{G}(w))^{\Gamma}\mathcal{H}(z,w)^{\Gamma}\right]\right\}\mathcal{G}_D(w,w)^{\mathrm{T}}$$

is analytic on the set $(\mathbb{C}\backslash[-\|X\|,\|X\|])^2$.

The rest of this section is devoted to the proof of Theorem 4. We start with two technical lemmas.

Lemma 7. Let E be a complex Banach space. Assume that $\Omega\subset\mathbb{C}$ is an open connected set and $f\colon\Omega\to E$ is a one-to-one analytic function such that $f'(z)\neq 0$ for all $z\in\Omega$. If $K\subset\Omega$ is a compact set, then $\sup\limits_{\substack{z_1,z_2\in K\\ z_1\neq z_2}}\dfrac{|z_2-z_1|}{\|f(z_2)-f(z_1)\|_E}$ is finite.

Proof. In order to reach contradiction, assume that the supremum is unbounded. In that case, there exist sequences $(p_n)_{n\geq 1},(q_n)_{n\geq 1}\subset K$ such that

$$\lim_{n\to\infty}\frac{\|f(p_n)-f(q_n)\|_E}{|p_n-q_n|}=0.$$

Since K is compact, by passing to a subsequence, we may assume that $(p_n,q_n)\to$

$(p, q) \in K^2$. If $p \neq q$, then

$$\lim_{n\to\infty} \frac{\|f(p_n) - f(q_n)\|_E}{|p_n - q_n|} = \frac{\|f(p) - f(q)\|_E}{|p - q|}.$$

Since f is one-to-one, the right hand side of the previous equation has to be strictly positive. This contradicts the fact that the left hand side of the previous equation is zero. If $p = q$, then the Taylor series expansion of f around p shows that $f'(p) = 0$. Since the latter equality contradicts our assumptions, we conclude that the supremum has to be bounded. □

Lemma 8. Let D be a dense subspace of a Banach space E and let $A : D \to \mathbb{C}^N$ be a linear operator. Assume that $\Omega \subset \mathbb{C}$ is an open connected set and $f : \Omega \to E$ is a one-to-one analytic function such that $f'(z) \neq 0$ for all $z \in \Omega$. If $S \subset \Omega$ is discrete in Ω, and $f(\Omega \setminus S) \subset D$, and $A \circ f : \Omega \setminus S \to \mathbb{C}^N$ is analytic, then $A \circ f$ extends analytically to Ω.

Proof. For notational simplicity, let $g = A \circ f : \Omega \setminus S \to \mathbb{C}^N$. Since S is discrete, it is enough to show that g can be analytically extended to a given point $\omega \in S$.

Since S is discrete, there exists $r \in (0, 1)$ such that $\{z \in \mathbb{C} : |z - w| \leq r\} \cap S = \{w\}$. Let $\gamma : \mathbb{R} \to \mathbb{C}$ be given by $\gamma(t) = w + re^{2\pi it}$. We start proving that A is Lipschitz on $f(\gamma(\mathbb{R}))$. Take $u_1, u_2 \in f(\gamma(\mathbb{R}))$. Since f is one-to-one, there exists a unique $z_k \in \gamma(\mathbb{R})$ such that $u_k = f(z_k)$ for $k = 1, 2$. It is easy to show that there exists $t_1 \in [0, 1)$ and $t_2 \geq t_1$ with $t_2 - t_1 \leq 1/2$ such that either $z_1 = \gamma(t_1)$ and $z_2 = \gamma(t_2)$ or $z_1 = \gamma(t_2)$ and $z_2 = \gamma(t_1)$. Without loss of generality, assume the former case. In this case,

$$\|g(z_2) - g(z_1)\|_{\mathbb{C}^N} = \|g(\gamma(t_2)) - g(\gamma(t_1))\|_{\mathbb{C}^N}$$

$$\leq \int_{t_1}^{t_2} |\gamma'(s)| \|g'(\gamma(s))\|_{\mathbb{C}^N} \mathrm{d}s$$

$$\leq 2\pi r m(t_2 - t_1),$$

where $m = \sup_{z \in \gamma(\mathbb{R})} \|g'(z)\|_{\mathbb{C}^N}$. By assumption, the function g is analytic on $\gamma(\mathbb{R}) \subset \Omega \setminus S$. Since $\gamma(\mathbb{R})$ is compact, we have that $m < \infty$. Observe that $|z_2 - z_1| = r|e^{2\pi i(t_2 - t_1)} - 1|$. Let $h : [0, 1/2] \to \mathbb{R}$ be given by $h(t) = |e^{2\pi i t} - 1|$. Observe that $h(0) = 0$ and $h(1/2) = 2$. A straightforward computation shows that $h''(t) = -\sqrt{2}\pi^2(1 - \cos(2\pi t))^{1/2}$ for all $t \in [0, 1/2]$. In particular, h is concave and therefore $h(t) \geq 4t$. As a consequence, $4r(t_2 - t_1) \leq |z_2 - z_1|$ and hence

$$\|g(z_2) - g(z_1)\|_{\mathbb{C}^N} \leq \frac{\pi}{2} m |z_2 - z_1|.$$

In particular, we have that

$$\|A(f(z_2)) - A(f(z_1))\|_{\mathbb{C}^N} \leq \frac{\pi}{2} m M \|f(z_2) - f(z_1)\|_E,$$

where $M = \sup_{\substack{z, z' \in \gamma(\mathbb{R}) \\ z \neq z'}} \frac{|z - z'|}{\|f(z) - f(z')\|_E}$. By the previous lemma, M is necessarily finite. Rewriting the previous equation we have that

$$\|A(u_2) - A(u_1)\|_{\mathbb{C}^N} \leq \frac{\pi}{2} m M \|u_2 - u_1\|_E,$$

and hence A is Lipschitz on $f(\gamma(\mathbb{R}))$.

Since f is analytic on Ω, Cauchy's integral formula, see, e.g., Theorem 3.31 in [46], implies that

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz.$$

Recall that since f is analytic in the norm topology, the integral also converges in norm. More specifically,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz = \int_0^1 f(w + re^{2\pi i t})\, dt = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f\left(w + re^{2\pi i \xi_j^{(n)}}\right)$$

in the norm topology of E, for a division of $[0,1]$ in n equal segments and a choice of $\xi_j^{(n)}$ in each segment. From the lipschitzianity of A on $f(\gamma(\mathbb{R}))$, we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (A \circ f)\left(w + re^{2\pi i \xi_j^{(n)}}\right) =: g(w)$$

exists. By Cauchy's theorem, $g(w)$ does not depend on the simple path taken. Furthermore, a similar lipschitzianity argument shows that $g(w_n) \to g(w)$ for all $w_n \to w$. By Morera's theorem, the extension $g : (\Omega \setminus S) \cup \{w\} \to \mathbb{C}^N$ is in fact analytic. □

Now we are in position to prove Theorem 4.

Proof of Theorem 4. Note that

$$\mathcal{G}_D(a, a^{\mathrm{T}}) \left\{ \Theta[\mathcal{H}(a,b)] + \Phi\left[\mathcal{H}(a,b)^\Gamma (\mathcal{G}(a) \otimes \mathcal{G}(b))^\Gamma \mathcal{H}(a,b)^\Gamma\right] \right\} \mathcal{G}_D(b, b^{\mathrm{T}})^{\mathrm{T}}$$

is well-defined on

$\{(a,b) \in M_d(\mathbb{C}) \times M_d(\mathbb{C}) \colon a - X, b - X$ invertible,

$$1 \notin \sigma(\Sigma\mathcal{G}(a) \otimes \mathcal{G}(b)^{\mathrm{T}}) \cup \sigma(\Sigma\mathcal{G}(a) \otimes \mathcal{G}(a)^{\mathrm{T}}) \cup \sigma(\Sigma\mathcal{G}(b) \otimes \mathcal{G}(b)^{\mathrm{T}})\},$$

where $\sigma(V)$ denotes the spectrum of the linear operator V. As a consequence, this allows us to extend $\Theta P((a-X)^{-1} \otimes (b-X)^{-1})$ to the same set.

Let $a, b \in \mathrm{M}_d$ be invertible. For such $a, b \in \mathrm{M}_d$, we define the map

$$\begin{aligned} g\colon (z,w) \mapsto \mathcal{G}_D((za)^{-1},(za)^{-\mathrm{T}}) \Big\{ & \Theta[\mathcal{H}((za)^{-1},(wb)^{-1})] + \\ & + \Phi\left[\mathcal{H}((za)^{-1},(wb)^{-1})^{\Gamma}(\mathcal{G}((za)^{-1}) \otimes \mathcal{G}((wb)^{-1}))^{\Gamma}\mathcal{H}((za)^{-1},(wb)^{-1})^{\Gamma}\right] \Big\} \times \\ & \times \mathcal{G}_D((wb)^{-1},(wb)^{-\mathrm{T}})^{\mathrm{T}}. \end{aligned}$$

By the discussion before Theorem 4, the mapping g is well-defined on a neighborhood of $(0,0)$. Let $g_w : z \mapsto g(z,w)$. For w_0 small enough, the mapping g_{w_0} is well-defined on

$$\begin{aligned} \{z \in \mathbb{C} : 1 - zaX \text{ invertible}, \\ 1 \notin \sigma(\Sigma\mathcal{G}((za)^{-1}) \otimes \mathcal{G}(b^{-1})^{\mathrm{T}}) \cup \sigma(\Sigma\mathcal{G}((za)^{-1}) \otimes \mathcal{G}((za)^{-1})^{\mathrm{T}})\}. \end{aligned}$$

We would like to be able to extend g_{w_0} to the possibly larger set Ω which is the connected component of $\{z \in \mathbb{C}\colon 1 - zaX \text{ invertible}\}$ containing zero. Observe that $z \mapsto \det(\mathrm{I}_{d^2} - \Sigma G((za)^{-1}) \otimes G(b^{-1})^{\mathrm{T}})$ maps $z = 0$ to one, so that the set of its zeros in Ω is necessarily discrete. Note that this holds as well for $z \mapsto \det(1_{d^2} - \Sigma G((za)^{-1}) \otimes G((za)^{-1})^{\mathrm{T}})$. Let us denote by S the union of these two discrete sets. The fact that g_{w_0} extends to S, i.e., that g_{w_0} has no singularities in Ω, is an immediate application of the previous lemma with $A(x) = P\left(x \otimes ((w_0 b)^{-1} - X)^{-1}\right)$, the analytic map $f(z) = ((za)^{-1} - X)^{-1} = z(1 - zaX)^{-1}a$, and the discrete set S defined above. Let z_0 be in the connected component of $\{z \in \mathbb{C} : 1 - zaX \text{ invertible}\}$ containing zero. By the above argument, the mapping $w \mapsto P\left(((z_0 a)^{-1} - X)^{-1} \otimes ((wb)^{-1} - X)^{-1}\right)$ is well-defined on a small disc around zero. Mutatis mutandis, we can extend this mapping to the set $\{w \in \mathbb{C} : 1 - wbX \text{ invertible}\}$. Thus, the map $(z, w) \mapsto P\left(((za)^{-1} - X)^{-1} \otimes ((wb)^{-1} - X)^{-1}\right)$ is analytic on the connected component $\{(z, w) \in \mathbb{C}^2 : 1 - zaX, 1 - wbX \text{ invertible}\}$ containing $(0, 0)$. $\square$

Chapter 4

Integral Representation of the Second-Order Cauchy Transform

In this chapter we provide a condition on a cluster function that ensures the integral representation of the second-order Cauchy transform. To prove this condition, we require some relations between the moments and the fluctuation moments of a measure that allow us to invoke results concerning the moment problem in the plane. In Section 4.1 we obtain these relations by comparing two types of two-variable Cauchy transforms. In Section 4.2 we present some inversion theorems for these Cauchy transforms. In particular, Theorem 6 will have interesting applications in the following chapter. Finally, in Section 4.3 we establish the aforementioned condition for the integral representation of the second-order Cauchy transform.

4.1 Two-Variables Cauchy Transforms and Their Moments

Ultimately, we will consider symmetric finite positive measures on $\mathbb{R}^2$ with compact support, as defined below. However, since most of the results hold in greater generality, we start with the rather general family of finite signed measures on $\mathbb{R}^2$. Consider the following two types of Cauchy transforms in two variables.

Definition 16. Let ν be a finite signed measure on $\mathbb{R}^2$. We define $\hat{G}, \tilde{G} : (\mathbb{C}\setminus\mathbb{R})^2 \to \mathbb{C}$ by

$$\hat{G}(z,w) = \int_{\mathbb{R}^2} \frac{1}{z-t_1}\frac{1}{w-t_2} \mathrm{d}\nu(t_1,t_2),$$
$$\tilde{G}(z,w) = \int_{\mathbb{R}^2} \frac{1}{z-t_1}\frac{1}{z-t_2}\frac{1}{w-t_1}\frac{1}{w-t_2} \mathrm{d}\nu(t_1,t_2).$$

We would like to point out that $\tilde{G}$ will be the prototype for the analytic counterpart of the second-order Cauchy transform. It is straightforward to verify that

$$\overline{\hat{G}(z,w)} = \hat{G}(\overline{z},\overline{w}) \quad \text{and} \quad \overline{\tilde{G}(z,w)} = \tilde{G}(\overline{z},\overline{w}).$$

A routine argument shows that both $\hat{G}$ and $\tilde{G}$ are analytic functions. Usually, it is easier to deal with $\hat{G}$ than with $\tilde{G}$. The following identity allows us to translate facts about $\hat{G}$ into facts about $\tilde{G}$.

Proposition 11. Let ν be a finite signed measure on $\mathbb{R}^2$. For every $z, w \in \mathbb{C}\setminus\mathbb{R}$,

$$(z-w)^2\tilde{G}(z,w) = \hat{G}(z,z) - \hat{G}(z,w) - \hat{G}(w,z) + \hat{G}(w,w). \tag{4.1}$$

Proof. Equation 4.1 trivially holds when $z = w$. If $z \neq w$, the following identity holds for all $t_1, t_2 \in \mathbb{R}$,

$$\frac{1}{z-t_1}\frac{1}{z-t_2}\frac{1}{w-t_1}\frac{1}{w-t_2} = \frac{(z-t_1)^{-1}-(w-t_1)^{-1}}{z-w}\frac{(z-t_2)^{-1}-(w-t_2)^{-1}}{z-w}.$$

Integrating both sides with respect to ν, the result follows. $\square$

In this context, in addition to the usual moments associated to a measure, we are interested in the so-called fluctuation moments.

Definition 17. Let ν be a finite signed measure on $\mathbb{R}^2$ with finite moments, i.e., for all $m, n \in \mathbb{N}$,

$$\int |t_1|^m |t_2|^n \mathrm{d}|\nu|(t_1, t_2) < \infty.$$

Its moments and fluctuation moments are defined by

$$M_{m,n} = \int_{\mathbb{R}^2} t_1^m t_2^n \mathrm{d}\nu(t_1, t_2) \quad \text{and} \quad \alpha_{m,n} = \int_{\mathbb{R}^2} \frac{t_1^m - t_2^m}{t_1 - t_2} \frac{t_1^n - t_2^n}{t_1 - t_2} \mathrm{d}\nu(t_1, t_2),$$

respectively.

By definition, we have that $\alpha_{m,0} = \alpha_{0,n} = 0$ for all $m, n \in \mathbb{N}$. Note that, for all $m, n \geq 1$,

$$\alpha_{m,n} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{i+j,(m-1)+(n-1)-(i+j)}. \tag{4.2}$$

For $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$, we let

$$B((x_0, y_0), r) = \{(x, y) \in \mathbb{R}^2 : \|(x, y) - (x_0, y_0)\| < r\},$$

where $\|(x, y)\| = \sqrt{x^2 + y^2}$. For a signed measure ν, we let $\mathrm{Supp}\,(\nu)$ denote the support of ν. The following proposition connects the moments and fluctuation moments with $\hat{G}$ and $\tilde{G}$, respectively. For notational convenience, we let $a \wedge b = \min(a, b)$.

Proposition 12. Let ν be a finite signed measure on $\mathbb{R}^2$. If ν is compactly supported,

then

$$\hat{G}(z,w) = \sum_{m,n\geq 0} \frac{M_{m,n}}{z^{m+1}w^{n+1}} \quad \text{and} \quad \tilde{G}(z,w) = \sum_{m,n\geq 0} \frac{\alpha_{m,n}}{z^{m+1}w^{n+1}}, \tag{4.3}$$

for every $z, w \in \mathbb{C} \setminus \mathbb{R}$ such that $B(0, |z| \wedge |w|) \supset \mathrm{Supp}\,(\nu)$.

Proof. Since $\mathrm{Supp}\,(\nu)$ is compact and $B(0, |z| \wedge |w|)$ is open, there exists $\epsilon > 0$ such that $\|s\|_2 < (1-\epsilon)(|z| \wedge |w|)$ for all $s \in \mathrm{Supp}\,(\nu)$. Thus, by the Tonelli-Fubini theorem,

$$\hat{G}(z,w) = \int_{\mathrm{Supp}(\nu)} \sum_{m,n\geq 0} \frac{t_1^m}{z^{m+1}} \frac{t_2^n}{w^{n+1}} \mathrm{d}\nu(t_1,t_2) = \sum_{m,n\geq 0} \frac{M_{m,n}}{z^{m+1}w^{n+1}}.$$

This establishes the first equality in (4.3).

Similarly, by the Tonelli-Fubini theorem,

$$\tilde{G}(z,w) = \int_{\mathrm{Supp}(\nu)} \sum_{i,j,k,l\geq 0} \frac{t_1^i}{z^{i+1}} \frac{t_2^j}{z^{j+1}} \frac{t_1^k}{w^{k+1}} \frac{t_2^l}{w^{l+1}} \mathrm{d}\nu(s,t) = \sum_{i,j,k,l\geq 0} \frac{M_{i+k,j+l}}{z^{i+j+2}w^{k+l+2}}.$$

By the Tonelli-Fubini theorem, we obtain that

$$\tilde{G}(z,w) = \sum_{a,b\geq 0} \frac{1}{z^{a+2}} \frac{1}{w^{b+2}} \sum_{i=0}^{a} \sum_{k=0}^{b} M_{i+k,a+b-(i+k)}.$$

By (4.2), we have that

$$\tilde{G}(z,w) = \sum_{a,b\geq 0} \frac{\alpha_{a+1,b+1}}{z^{a+2}w^{b+2}} = \sum_{m,n\geq 1} \frac{\alpha_{m,n}}{z^{m+1}w^{n+1}}.$$

Since $\alpha_{m,0} = \alpha_{0,n} = 0$ for all $m, n \in \mathbb{N}$, the second equality in (4.3) follows. □

The previous proposition and (4.1) recover the following identity in [30].

Corollary 3. Let ν be a finite signed measure on $\mathbb{R}^2$. If ν is compactly supported,

then

$$\sum_{m,n\geq 1} z^{m-1}w^{n-1}\alpha_{m,n} = \sum_{m,n\geq 1} \frac{z^m - w^m}{z-w}\frac{z^n - w^n}{z-w} M_{m-1,n-1}, \tag{4.4}$$

for all $z, w \in \mathbb{C} \setminus \mathbb{R}$ such that $z \neq w$ and $B(0, |z|^{-1} \wedge |w|^{-1}) \supset \mathrm{Supp}\,(\nu)$.

Proof. By the second equality in (4.3), the left hand side of equation (4.4) equals

$$\frac{1}{(zw)^2}\tilde{G}(z^{-1}, w^{-1}).$$

By (4.1) and the first equality in (4.3), the latter equals

$$\frac{1}{(z-w)^2}\sum_{m,n\geq 0} M_{m,n}\left(z^{m+1}z^{n+1} - z^{m+1}w^{n+1} - w^{m+1}z^{n+1} + w^{m+1}w^{n+1}\right).$$

Therefore,

$$\frac{1}{(zw)^2}\tilde{G}(z^{-1}, w^{-1}) = \sum_{m,n\geq 0} \frac{z^{m+1} - w^{m+1}}{z-w}\frac{z^{n+1} - w^{n+1}}{z-w}\int_{\mathbb{R}^2} s^m t^n \mathrm{d}\nu(s,t).$$

The result follows. □

Given a measure ν on $\mathbb{R}^2$, we let ν^T be the measure determined by $\nu^T([a,b]\times[c,d]) = \nu([c,d]\times[a,b])$. We say that a finite signed measure ν is symmetric if $\nu = \nu^T$. It is not hard to verify that if ν is symmetric then $\hat{G}(z,w) = \hat{G}(w,z)$. (The converse is an easy consequence of Theorem 7 in Section 4.2.)

In (4.2) we established an expression for the fluctuation moments in terms of the moments. The following theorem *inverts* that relation when the underlying measure is symmetric.

Theorem 5. Let ν be a symmetric finite signed measure on $\mathbb{R}^2$ with finite moments.

Then, for all $m, n \in \mathbb{N}$,

$$M_{m,n} = \frac{2\alpha_{m+1,n+1} - \alpha_{m+2,n} - \alpha_{m,n+2}}{2}. \tag{4.5}$$

Before proving this theorem, we establish the following lemma.

Lemma 9. Let ν be a finite signed measure on $\mathbb{R}^2$. If ν is compactly supported, then

$$\hat{G}(z, z) = \sum_{m \geq 0} \frac{\alpha_{m,1}}{z^{m+1}},$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ such that $B(0, |z|) \supset \mathrm{Supp}\,(\nu)$.

Proof. By Proposition 12, we have that

$$\hat{G}(z, z) = \sum_{i,j \geq 0} \frac{M_{i,j}}{z^{i+j+2}} = \sum_{p \geq 0} \frac{1}{z^{p+2}} \sum_{i=0}^{p} M_{i,p-i}.$$

Recall the relation between moments and fluctuation moments in (4.2). Thus,

$$\hat{G}(z, z) = \sum_{p \geq 0} \frac{\alpha_{p+1,1}}{z^{p+2}} = \sum_{m \geq 1} \frac{\alpha_{m,1}}{z^{m+1}}.$$

Since $\alpha_{0,1} = 0$, the result follows. □

Proof of Theorem 5. Assume first that ν is compactly supported. Since ν is symmetric, (4.1) implies that

$$\hat{G}(z, w) = \frac{\hat{G}(z, z) + \hat{G}(w, w) - (z - w)^2 \tilde{G}(z, w)}{2}. \tag{4.6}$$

Let $z, w \in \mathbb{C} \setminus \mathbb{R}$ be such that $B(0, |z| \wedge |w|) \supset \mathrm{Supp}\,(\nu)$. Then, by Proposition 12,

$$\begin{aligned}
(z-w)^2 \tilde{G}(z,w) &= (z^2 - 2zw + w^2) \sum_{i,j \geq 1} \frac{\alpha_{i,j}}{z^{i+1} w^{j+1}} \\
&= \sum_{i,j \geq 1} \frac{\alpha_{i,j}}{z^{i-1} w^{j+1}} - 2 \sum_{i,j \geq 1} \frac{\alpha_{i,j}}{z^i w^j} + \sum_{i,j \geq 1} \frac{\alpha_{i,j}}{z^{i+1} w^{j-1}} \\
&= \sum_{\substack{m \geq -1 \\ n \geq 0}} \frac{\alpha_{m+2,n}}{z^{m+1} w^{n+1}} - 2 \sum_{m,n \geq 0} \frac{\alpha_{m+1,n+1}}{z^{m+1} w^{n+1}} + \sum_{\substack{m \geq 0 \\ n \geq -1}} \frac{\alpha_{m,n+2}}{z^{m+1} w^{n+1}} \\
&= \sum_{m,n \geq 0} \frac{\alpha_{m+2,n} + \alpha_{m,n+2} - 2\alpha_{m+1,n+1}}{z^{m+1} w^{n+1}} + \sum_{n \geq 0} \frac{\alpha_{1,n}}{w^{n+1}} + \sum_{m \geq 0} \frac{\alpha_{m,1}}{z^{m+1}}.
\end{aligned}$$

Since ν is symmetric, $\alpha_{m,n} = \alpha_{n,m}$ for all $m, n \in \mathbb{N}$. Hence, by the previous lemma,

$$(z-w)^2 \tilde{G}(z,w) = \sum_{m,n \geq 0} \frac{\alpha_{m+2,n} + \alpha_{m,n+2} - 2\alpha_{m+1,n+1}}{z^{m+1} w^{n+1}} + \hat{G}(w,w) + \hat{G}(z,z).$$

By (4.6), we obtain that

$$\hat{G}(z,w) = \sum_{m,n \geq 0} \frac{2\alpha_{m+1,n+1} - \alpha_{m+2,n} - \alpha_{m,n+2}}{2} \frac{1}{z^{m+1} w^{n+1}}.$$

By Proposition 12 we also have that $\hat{G}(z,w) = \sum_{m,n \geq 0} \frac{M_{m,n}}{z^{m+1} w^{n+1}}$. By the uniqueness of the power series expansion of an analytic function, we conclude that, for all $m, n \geq 0$,

$$M_{m,n} = \frac{2\alpha_{m+1,n+1} - \alpha_{m+2,n} - \alpha_{m,n+2}}{2}.$$

Assume now that ν has unbounded support. For $L > 0$, let ν^L be the finite signed measure defined by

$$\nu^L(A) = \nu(A \cap [-L, L] \times [-L, L]).$$

Let $(M^L_{p,q} : p, q \in \mathbb{N})$ and $(\alpha^L_{p,q} : p, q \in \mathbb{N})$ be the moments and fluctuation moments of ν^L, respectively. By the first part of this proof, for all $m, n \in \mathbb{N}$,

$$M^L_{m,n} = \frac{2\alpha^L_{m+1,n+1} - \alpha^L_{m+2,n} - \alpha^L_{m,n+2}}{2}.$$

Since ν has finite moments, the dominated convergence theorem implies that

$$\begin{aligned} M_{m,n} &= \lim_{L\to\infty} M^L_{m,n} \\ &= \lim_{L\to\infty} \frac{2\alpha^L_{m+1,n+1} - \alpha^L_{m+2,n} - \alpha^L_{m,n+2}}{2} \\ &= \frac{2\alpha_{m+1,n+1} - \alpha_{m+2,n} - \alpha_{m,n+2}}{2}, \end{aligned}$$

as we wanted to prove. □

4.2 Inversion Theorems

In this section we present some inversion theorems for $\hat{G}$ and $\tilde{G}$.

In the next chapter, the following theorem will provide an integral representation for the asymptotic covariance of the linear statistics of random matrices.

Theorem 6. Let ν be a finite signed measure on $\mathbb{R}^2$ with compact support, i.e., $\operatorname{Supp}(\nu) \subset [-M, M]^2$ for some $M > 0$. Let $\Omega \subset \mathbb{C}$ be a domain containing $[-M, M]$. If $f, g : \Omega \to \mathbb{C}$ are analytic functions and

$$\tilde{G}(z, w) = \int_{\mathbb{R}^2} \frac{1}{z - t_1} \frac{1}{z - t_2} \frac{1}{w - t_1} \frac{1}{w - t_2} \mathrm{d}\nu(t_1, t_2),$$

then

$$\frac{1}{(2\pi i)^2}\int_{\mathcal{C}}\int_{\mathcal{C}} f(z)g(w)\tilde{G}(z,w)\mathrm{d}z\mathrm{d}w = \int_{\mathbb{R}^2} \frac{f(t_1)-f(t_2)}{t_1-t_2}\frac{g(t_1)-g(t_2)}{t_1-t_2}\mathrm{d}\nu(t_1,t_2), \quad (4.7)$$

where $\mathcal{C} \subset \Omega$ is a positively oriented simple closed contour enclosing $[-M, M]$.

Proof. Let I be the left hand side of (4.7). By assumption, we have that $\operatorname{Supp}(\nu) \subset [-M, M]^2$. Thus,

$$I = \frac{1}{(2\pi i)^2}\int_{\mathcal{C}}\int_{\mathcal{C}} f(z)g(w)\int_{[-M,M]^2} \frac{1}{z-t_1}\frac{1}{z-t_2}\frac{1}{w-t_1}\frac{1}{w-t_2}\mathrm{d}\nu(t_1,t_2)\mathrm{d}z\mathrm{d}w.$$

Since ν is a finite measure, a routine application of the Tonelli-Fubini theorem shows that

$$I = \int_{[-M,M]^2}\left(\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f(z)\ \mathrm{d}z}{(z-t_1)(z-t_2)}\right)\left(\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{g(w)\ \mathrm{d}w}{(w-t_1)(w-t_2)}\right)\mathrm{d}\nu(t_1,t_2).$$

Observe that

$$\frac{f(z)}{(z-t_1)(z-t_2)} = \frac{1}{t_1-t_2}\left(\frac{f(z)}{z-t_1} - \frac{f(z)}{z-t_2}\right).$$

By Cauchy's integral formula, we conclude that

$$I = \int_{[-M,M]^2} \frac{f(t_1)-f(t_2)}{t_1-t_2}\frac{g(t_1)-g(t_2)}{t_1-t_2}\mathrm{d}\nu(t_1,t_2),$$

as required. □

We now turn to inversion theorems in the spirit of the Stieltjes inversion formula.

To this end, we define

$$\hat{H}(z,w) = \hat{G}(z,w) - \hat{G}(z,\overline{w}) - \hat{G}(\overline{z},w) + \hat{G}(\overline{z},\overline{w}),$$
$$\tilde{H}(z,w) = -\frac{(z-w)^2\tilde{G}(z,w) - (z-\overline{w})^2\tilde{G}(z,\overline{w}) - (\overline{z}-w)^2\tilde{G}(\overline{z},w) + (\overline{z}-\overline{w})^2\tilde{G}(\overline{z},\overline{w})}{2}.$$

As $\hat{G}$ and $\tilde{G}$ are related by (4.1), $\hat{H}$ and $\tilde{H}$ are related by the following.

Lemma 10. Let ν be a finite signed measure. Then $\tilde{H}(z,w) = \dfrac{\hat{H}(z,w) + \hat{H}(w,z)}{2}$ for all $z, w \in \mathbb{C} \setminus \mathbb{R}$.

Proof. By equation (4.1) we have that

$$\begin{aligned}
-(z-w)^2\tilde{G}(z,w) &= -\hat{G}(z,z) + \hat{G}(z,w) + \hat{G}(w,z) - \hat{G}(w,w),\\
(z-\overline{w})^2\tilde{G}(z,\overline{w}) &= +\hat{G}(z,z) - \hat{G}(z,\overline{w}) - \hat{G}(\overline{w},z) + \hat{G}(\overline{w},\overline{w}),\\
(\overline{z}-w)^2\tilde{G}(\overline{z},w) &= +\hat{G}(\overline{z},\overline{z}) - \hat{G}(\overline{z},w) - \hat{G}(w,\overline{z}) + \hat{G}(w,w),\\
-(\overline{z}-\overline{w})^2\tilde{G}(\overline{z},\overline{w}) &= -\hat{G}(\overline{z},\overline{z}) + \hat{G}(\overline{z},\overline{w}) + \hat{G}(\overline{w},\overline{z}) - \hat{G}(\overline{w},\overline{w}).
\end{aligned}$$

Adding the previous equations, the result follows. □

The following theorem, communicated to the author by Mingo [30], allows us to recover ν from $\hat{G}$. In particular, it shows that $\hat{G}_\nu$ characterizes ν within the set of finite signed measures. We will use the following notation, $x_\epsilon = x + i\epsilon$ and $y_\delta = y + i\delta$ for all $x, y \in \mathbb{R}$ and $\epsilon, \delta > 0$.

Theorem 7. Let ν be a finite signed measure on $\mathbb{R}^2$. Then, for every $a < b$ and $c < d$,

$$\nu(A_0) - \frac{1}{2}\nu(A_1) + \frac{1}{4}\nu(A_2) = \frac{1}{(2\pi i)^2}\lim_{\epsilon,\delta\to 0^+}\int_a^b\int_c^d \hat{H}(x_\epsilon, y_\delta)\mathrm{d}x\mathrm{d}y, \tag{4.8}$$

where $A_0 = [a,b] \times [c,d]$, $A_1 = [a,b] \times \{c,d\} \cup \{a,b\} \times [c,d]$, and $A_2 = \{a,b\} \times \{c,d\}$.

Observe that A_0 is the square $[a,b] \times [c,d]$, A_1 is the border of A_0, and A_2 is the set containing the corners of A_0.

Proof. Since $\dfrac{1}{y_\delta - t_2} - \dfrac{1}{\overline{y_\delta} - t_2} = \dfrac{-2i\delta}{(y-t_2)^2+\delta^2}$, we have that

$$\begin{aligned} \hat{G}(x_\epsilon, y_\delta) - \hat{G}(x_\epsilon, \overline{y_\delta}) &= \int_{\mathbb{R}^2} \frac{1}{x_\epsilon - t_1} \frac{-2i\delta}{(y-t_2)^2+\delta^2} \mathrm{d}\nu(t_1,t_2), \\ \hat{G}(\overline{x_\epsilon}, y_\delta) - \hat{G}(\overline{x_\epsilon}, \overline{y_\delta}) &= \int_{\mathbb{R}^2} \frac{1}{\overline{x_\epsilon} - t_1} \frac{-2i\delta}{(y-t_2)^2+\delta^2} \mathrm{d}\nu(t_1,t_2). \end{aligned}$$

Mutatis mutandis, we obtain that

$$\hat{H}(x_\epsilon, y_\delta) = -4 \int_{\mathbb{R}^2} \frac{\epsilon}{(x-t_1)^2+\epsilon^2} \frac{\delta}{(y-t_2)^2+\delta^2} \mathrm{d}\nu(t_1,t_2). \tag{4.9}$$

Since ν is finite, the Tonelli-Fubini theorem readily implies that

$$\int_a^b \int_c^d \hat{H}(x_\epsilon, y_\delta) \mathrm{d}x \mathrm{d}y = -4 \int_{\mathbb{R}^2} \chi_{a,b}(t_1,\epsilon) \chi_{c,d}(t_2,\delta) \mathrm{d}\nu(t_1,t_2),$$

where

$$\chi_{a,b}(t_1,\epsilon) = \int_a^b \frac{\epsilon}{(x-t_1)^2+\epsilon^2} \mathrm{d}x \quad \text{and} \quad \chi_{c,d}(t_2,\delta) = \int_c^d \frac{\delta}{(y-t_2)^2+\delta^2} \mathrm{d}y.$$

A change of variables shows that

$$\chi_{p,q}(t,\gamma) = \int_{\frac{p-t}{\gamma}}^{\frac{q-t}{\gamma}} \frac{1}{1+u^2} \mathrm{d}u.$$

In particular, we have that $0 \leq \chi_{a,b}(s,\epsilon), \chi_{c,d}(t,\delta) \leq \pi$. It is not hard to verify that

$$\lim_{\gamma \to 0^+} \chi_{p,q}(t,\gamma) = \pi \left[1_{[p,q]}(t) - \frac{1}{2} 1_{\{p,q\}}(t) \right],$$

where $1_A(\cdot)$ is the indicator function of the set A. Since ν is finite, the dominated convergence theorem implies that

$$\frac{1}{(2\pi i)^2} \lim_{\epsilon,\delta \to 0^+} \int_a^b \int_c^d \hat{H}(x_\epsilon, y_\delta) \mathrm{d}x \mathrm{d}y$$

equals

$$\int_{\mathbb{R}^2} \left[1_{[a,b]}(t_1) - \frac{1}{2} 1_{\{a,b\}}(t_1) \right] \left[1_{[c,d]}(t_2) - \frac{1}{2} 1_{\{c,d\}}(t_2) \right] \mathrm{d}\nu(t_1, t_2).$$

Since the latter equals $\nu(A_0) - \frac{1}{2}\nu(A_1) + \frac{1}{4}\nu(A_2)$, the result follows. □

From the previous theorem and Lemma 10, we obtain a similar inversion formula for $\tilde{G}$. Our proof for the next theorem differs from that in [30], which does not use the relation between $\hat{H}$ and $\tilde{H}$. For a given measure ν, we let $\nu^S = (\nu + \nu^T)/2$.

Theorem 8. Let ν be a finite signed measure on $\mathbb{R}^2$. Then, for every $a < b$ and $c < d$,

$$\nu^S(A_0) - \frac{1}{2}\nu^S(A_1) + \frac{1}{4}\nu^S(A_2) = \frac{1}{(2\pi i)^2} \lim_{\epsilon,\delta \to 0^+} \int_a^b \int_c^d \tilde{H}(x_\epsilon, y_\delta) \mathrm{d}x \mathrm{d}y$$

where $A_0 = [a,b] \times [c,d]$, $A_1 = [a,b] \times \{c,d\} \cup \{a,b\} \times [c,d]$, and $A_2 = \{a,b\} \times \{c,d\}$.

In particular, $\tilde{G}_\nu$ characterizes ν within the set of symmetric finite signed measures.

Proof. It is straightforward to verify that $\hat{G}_{\nu^T}(z,w) = \hat{G}_\nu(w,z)$ and $\hat{H}_{\nu^T}(z,w) = \hat{H}_\nu(w,z)$ for all $z,w \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 10, $\tilde{H}(z,w) = (\hat{H}(z,w) + \hat{H}(w,z))/2$.

Therefore,

$$\tilde{H}_\nu(z,w) = \frac{\hat{H}_\nu(z,w) + \hat{H}_{\nu^T}(z,w)}{2}.$$

By Theorem 7, we conclude that

$$\frac{1}{(2\pi i)^2} \lim_{\epsilon,\delta\to 0^+} \int_a^b \int_c^d \tilde{H}(x_\epsilon, y_\delta)\mathrm{d}x\mathrm{d}y = \nu^S(A_0) - \frac{1}{2}\nu^S(A_1) + \frac{1}{4}\nu^S(A_2),$$

as required. □

4.3 Integral Representation of the Second-Order Cauchy Transform

Our starting point is a criterion for the solvability of the Hamburger moment problem in $\mathbb{R}^2$. Given a sequence $(M_{m,n} : m, n \in \mathbb{N}) \subset \mathbb{R}$, consider the mapping

$$\mathbb{R}[x,y] \ni P(x,y) = \sum_{m,n=0}^{d} c_{m,n} x^m y^n \mapsto \sum_{m,n=0}^{d} c_{m,n} M_{m,n} =: P_M \in \mathbb{R}.$$

If $P_M \geq 0$ for all $P \in \mathbb{R}[x,y]$ such that $P(x,y) \geq 0$ for all $x, y \in \mathbb{R}$, we say that $(M_{m,n} : m, n \in \mathbb{N})$ is non-negative.

Theorem 9 ([21])**.** Assume that $(M_{m,n} : m, n \in \mathbb{N})$ is given. For the existence of a positive measure ν such that, for all $m, n \in \mathbb{N}$,

$$\int_{\mathbb{R}^2} t_1^m t_2^n \mathrm{d}\nu(t_1, t_2) = M_{m,n},$$

it is necessary and sufficient that $(M_{m,n} : m, n \in \mathbb{N})$ be non-negative.

The previous theorem leads to the promised criterion regarding the integral representation of the second-order Cauchy transform. Given an $N \times N$ selfadjoint random matrix X_N with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$, the correlation function $R_l^{(N)} : \mathbb{R}^l \to \mathbb{R}$ is defined as the function satisfying

$$\sum_{\substack{j_1,\cdots,j_l \\ j_s \neq j_t (s\neq t)}} \mathbb{E}\left(f(\lambda_{j_1}, \ldots, \lambda_{j_l})\right) = \int_{\mathbb{R}^l} f(x_1, \ldots, x_l) R_l^{(N)}(x_1, \ldots, x_l) \mathrm{d}x_1 \cdots \mathrm{d}x_l,$$

for all continuous and bounded functions $f : \mathbb{R}^l \to \mathbb{R}$. With this notation, the cluster function $T_2^{(N)} : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$T_2^{(N)}(x, y) = R_1^{(N)}(x) R_1^{(N)}(y) - R_2^{(N)}(x, y), \tag{4.10}$$

for all $x, y \in \mathbb{R}$. For a more detailed discussion about correlation and cluster functions, we refer the reader to Section 1.1.2 in [41].

Theorem 10. Assume that $(X_N : N \in \mathbb{N})$ is a selfadjoint random matrix ensemble such that

1) For all $m, n \in \mathbb{N}$, the following limits exist

$$\alpha_n = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left(\mathrm{Tr}\left(X_N^n\right)\right) \quad \text{and} \quad \alpha_{m,n} = \lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left(X_N^m\right), \mathrm{Tr}\left(X_N^n\right)\right);$$

2) There exist $M > 0$ and $K > 0$ such that $\limsup_{m,n\to\infty} |\alpha_{m,n}|^{1/(m+n)} \leq M$ and, for all $m, n \in \mathbb{N}$,

$$\left| \frac{2\alpha_{m+1,n+1} - \alpha_{m+2,n} - \alpha_{m,n+2}}{2} \right| \leq K M^{m+n}.$$

If $T_2^{(N)}(x, y)$ is non-negative for all $N \in \mathbb{N}$ and $x, y \in \mathbb{R}$, then there exists a symmetric positive measure ν such that $\mathrm{Supp}\,(\nu) \subset [-M, M]^2$ and, for all $z, w \in \mathbb{C}$ with

$|z|, |w| > M$,

$$G_2(z,w) = \int_{\mathbb{R}^2} \frac{1}{z-t_1}\frac{1}{z-t_2}\frac{1}{w-t_1}\frac{1}{w-t_2}\mathrm{d}\nu(t_1,t_2). \tag{4.11}$$

Recall that the second-order Cauchy transform is defined as the generating function of the second-order moments, i.e.,

$$G_2(z,w) = \sum_{m,n\geq 0} \frac{\alpha_{m,n}}{z^{m+1}w^{n+1}}.$$

Assumptions 1) and 2) imply that this generating function is well-defined and that it converges absolutely in the set $\{(z,w) \in \mathbb{C}^2 : |z|, |w| > M\}$. It is important to remark that in some cases $T_2^{(N)}(x,y)$ admits a determinantal formula, see, e.g., Propoposition 4.2.4 in [41]. In this case, $T_2^{(N)}(x,y) = K_N(x,y)^2$ for some function $K_N : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, which clearly implies the positivity assumed in the previous theorem.

Another possibility is to represent the second-order Cauchy transform as in (4.11) but using a finite *signed* measure instead of a positive one. This extra flexibility may lead to more general criteria applicable to a wider range of random matrix ensembles. This possibility is left to future investigations.

Proof of Theorem 10. For all $N \in \mathbb{N}$ and $m, n \in \mathbb{N}$, we define $M_{m,n}^{(N)}$ as

$$\frac{2\mathrm{Cov}\left(\mathrm{Tr}(X_N^{m+1}), \mathrm{Tr}(X_N^{n+1})\right) - \mathrm{Cov}\left(\mathrm{Tr}(X_N^{m+2}), \mathrm{Tr}(X_N^{n})\right) - \mathrm{Cov}\left(\mathrm{Tr}(X_N^{m}), \mathrm{Tr}(X_N^{n+2})\right)}{2}.$$

By Proposition 1.1.6 in [41], we have that $M_{m,n}^{(N)}$ equals the integral of

$$\frac{2(x^{m+1}-y^{m+1})(x^{n+1}-y^{n+1}) - (x^{m+2}-y^{m+2})(x^n-y^n) - (x^m-y^m)(x^{n+2}-y^{n+2})}{4}$$

with respect to $T_2^{(N)}(x,y)\mathrm{d}x\mathrm{d}y$. A straightforward manipulation shows that

$$M_{m,n}^{(N)} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{x^m y^n + x^n y^m}{2}\frac{(x-y)^2}{2} T_2^{(N)}(x,y)\mathrm{d}x\mathrm{d}y.$$

By definition, i.e., (4.10), the function $T_2^{(N)}$ is symmetric. Therefore,

$$M_{m,n}^{(N)} = \int_{\mathbb{R}}\int_{\mathbb{R}} x^m y^n \frac{(x-y)^2}{2} T_2^{(N)}(x,y)\mathrm{d}x\mathrm{d}y.$$

By our assumption, $\frac{(x-y)^2}{2}T_2^{(N)}(x,y)$ defines a positive measure. In particular, this shows that sequence $\left(M_{m,n}^{(N)} : m,n \in \mathbb{N}\right)$ is non-negative for all $N \in \mathbb{N}$.

Note that, for all $m,n \in \mathbb{N}$,

$$\lim_{N\to\infty} M_{m,n}^{(N)} = \frac{2\alpha_{m+1,n+1} - \alpha_{m+2,n} - \alpha_{m,n+2}}{2} =: M_{m,n}. \tag{4.12}$$

Since $\left(M_{m,n}^{(N)} : m,n \in \mathbb{N}\right)$ is non-negative for all $N \in \mathbb{N}$, we conclude that $(M_{m,n} : m,n \in \mathbb{N})$ is also non-negative. Therefore, there exists a measure ν_0 with $(M_{m,n} : m,n \in \mathbb{N})$ as its moments. Note that, for all $m,n \in \mathbb{N}$, we have that $M_{m,n} = M_{n,m}$. Hence, $\nu = \nu_0^S$ is a symmetric positive measure with the same moments as ν_0. By plugging (4.12) in (4.2), it is straightforward to verify that $(\alpha_{m,n} : m,n \in \mathbb{N})$ are the fluctuation moments of ν. Note that 2) implies that $\mathrm{Supp}\,(\nu) \subset [-M,M]^2$. Proposition 12 then implies that

$$G_2(z,w) = \sum_{m,n\geq 0} \frac{\alpha_{m,n}}{z^{m+1}w^{n+1}} = \tilde{G}(z,w),$$

for all $|z|,|w| > M$. The result follows. □

Chapter 5

Convergence of the Covariance of Resolvents to the Second-Order Cauchy Transform

In this last chapter we address the third problem discussed in the introduction, namely, to find conditions that guarantee the convergence of the covariance of resolvents to the second-order Cauchy transform. We propose two conditions. The first one is a weaker form of a large deviation principle for the norm of a random matrix; while the second one is a Poincaré-type inequality for the linear statistics of a random matrix. Recall that a linear statistic is a random variable of the form $\mathrm{Tr}\,(f(X_N))$ where $(X_N : N \in \mathbb{N})$ is a random matrix ensemble and f is a function; if f is analytic (resp. continuously differentiable), we say that $\mathrm{Tr}\,(f(X_N))$ is an analytic (resp. continuously differentiable) linear statistic.

The aforementioned conditions not only ensure the convergence of the covariance of resolvents, but allow us to compute the covariance of analytic linear statistics using the second-order Cauchy transform. When combined with the notion of second-order limit distribution, these conditions also lead to the asymptotic Gaussianity of the continuously differentiable linear statistics. In Section 5.1 we state our main results. In Section 5.2 we provide examples of random matrix ensembles satisfying the aforementioned conditions. In particular, we show that block Gaussian matrices belong to this class of ensembles. All proofs are deferred to Section 5.3.

5.1 Main Results

Let us recall the following definition from [12].

Definition 18. Let $(X_N : N \in \mathbb{N})$ be a random matrix ensemble. We say that it has a second-order limit distribution if

i) For all $m, n \in \mathbb{N}$, the following limits exist

$$\alpha_n = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\left(X_N^n\right)\right) \quad \text{and} \quad \alpha_{m,n} = \lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left(X_N^m\right), \mathrm{Tr}\left(X_N^n\right)\right).$$

ii) For all $r \geq 3$ and all $n_1, \ldots, n_r \in \mathbb{N}$,

$$\lim_{N\to\infty} k_r\left(\mathrm{Tr}\left(X_N^{n_1}\right), \ldots, \mathrm{Tr}\left(X_N^{n_r}\right)\right) = 0.$$

Recall that the collections $(\alpha_n : n \in \mathbb{N})$ and $(\alpha_{m,n} : m, n \in \mathbb{N})$ are called the first- and second-order moments of $(X_N : N \in \mathbb{N})$, respectively. Also, recall that the second-order Cauchy transform is defined as the generating function of the second-order moments, i.e.,

$$G_2(z, w) = \sum_{m,n\geq 0} \frac{\alpha_{m,n}}{z^{m+1}w^{n+1}}.$$

In order to have a well defined second-order Cauchy transform, we always implicitly assume that

A0. Both the first- and second-order moments exists.

Clearly, A0 corresponds to Part i) in Definition 18. In addition to A0, we also consider the following two assumptions:

A1. There exists $M > 0$ such that

$$\lim_{N\to\infty} N^8 \mathrm{P}\left(\|X_N\| > M\right) = 0.$$

A2. There exists $K > 0$ such that, for all continuously differentiable $f, g \in C^1(\mathbb{R})$ and $N \in \mathbb{N}$,

$$\left|\mathrm{Cov}\left(\mathrm{Tr}\left(f(X_N)\right), \mathrm{Tr}\left(g(X_N)\right)\right)\right| \leq K\|f'\|_\infty \|g'\|_\infty.$$

Note that A1 is a weaker form of a large deviation principle. In fact, in Section 5.2, we use the fact that the largest eigenvalue of a GUE matrix satisfies a large deviation principle to prove that block Gaussian matrices satisfy A1. Note that A2 is equivalent to the following:

A2. There exists $K > 0$ such that, for all $f \in C^1(\mathbb{R})$ and $N \in \mathbb{N}$,

$$\mathrm{Var}\left(\mathrm{Tr}\left(f(X_N)\right)\right) \leq K\|f'\|_\infty^2.$$

This last inequality is common in the literature, as evidenced in Section 5.2. In fact, it can be seen as a matricial version of the Poincaré inequality, see Theorem 4.1 in [18] and references therein.

Assumptions A1 and A2 have immediate consequences for the behavior of the covariance of linear statistics. By abuse of notation, we denote by $f|_M$ the restriction of $f : \mathbb{R} \to \mathbb{C}$ to the interval $[-M, M]$. For $f \in C^1([-M,M])$, we let $\|f\|_M = \sup\{|f(x)| : x \in [-M, M]\}$.

Proposition 13. If $(X_N : N \in \mathbb{N})$ is a selfadjoint random matrix ensemble satisfying A1 and A2, then there exists a bilinear functional $\rho : C^1([-M,M]) \times C^1([-M,M]) \to \mathbb{C}$ such that

a) For all $f, g \in C^1([-M, M])$,

$$|\rho(f,g)| \leq K\|f'\|_M \|g'\|_M;$$

b) For all $f, g : \mathbb{R} \to \mathbb{C}$ polynomially bounded with $f|_M, g|_M \in C^1([-M, M])$,

$$\lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left(f(X_N)\right), \mathrm{Tr}\left(g(X_N)\right)\right) = \rho(f|_M, g|_M). \tag{5.1}$$

Note that Property a) implies that ρ is continuous with respect to the C^1-norm. Property b) says, in particular, that the limit in (5.1) exists. As an easy consequence to the previous proposition, we have the following.

Corollary 4. If $(X_N : N \in \mathbb{N})$ is a selfadjoint random matrix ensemble satisfying A1 and A2, then, for all $m, n \geq 1$,

$$|\alpha_{m,n}| \leq KmnM^{m+n-2}.$$

In particular, G_2 determines an analytic function on $\{(z, w) \in \mathbb{C}^2 : |z|, |w| > M\}$.

The next theorem exhibits a deeper implication of our standing assumptions on the analyticity of the second-order Cauchy transform. Specifically, it establishes that

G_2 can be analytically extended to $(\mathbb{C} \setminus [-M, M])^2$. In addition, it addresses the question in the introduction regarding the convergence of the covariance of resolvents.

Theorem 11. If $(X_N : N \in \mathbb{N})$ is a selfadjoint random matrix ensemble satisfying A1 and A2, then G_2 can be analytically extended to $(\mathbb{C} \setminus [-M, M])^2$ and, for all $z, w \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{N\to\infty} \operatorname{Cov}\left(\operatorname{Tr}\left((z - X_N)^{-1}\right), \operatorname{Tr}\left((w - X_N)^{-1}\right)\right) = G_2(z, w).$$

Note that Proposition 13 does not address the question of finding the actual values that ρ takes. The next theorem makes explicit the usefulness of the second-order Cauchy transform in this respect.

Theorem 12. Assume that $f, g : \mathbb{R} \to \mathbb{C}$ are polynomially bounded functions such that $f|_M$ and $g|_M$ extend analytically to a domain $\Omega \supset [-M, M]$. If $(X_N : N \in \mathbb{N})$ is a selfadjoint random matrix ensemble satisfying A1 and A2, then

$$\lim_{N\to\infty} \operatorname{Cov}\left(\operatorname{Tr}\left(f(X_N)\right), \operatorname{Tr}\left(g(X_N)\right)\right) = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}} \int_{\mathcal{C}} f(z) g(w) G_2(z, w) \mathrm{d}z \mathrm{d}w, \quad (5.2)$$

where $\mathcal{C} \subset \Omega$ is a positively oriented simple closed contour enclosing $[-M, M]$.

Recall that in the previous chapter we pursued an integral representation for G_2 of the form

$$G_2(z, w) = \int_{\mathbb{R}^2} \frac{1}{z - t_1} \frac{1}{z - t_2} \frac{1}{w - t_1} \frac{1}{w - t_2} \mathrm{d}\nu(t_1, t_2), \quad (5.3)$$

where ν is a finite measure supported on $[-M, M]^2$. Conditioned on this integral representation, the previous theorem and Theorem 6 lead to the following presentation for the covariance of linear statistics, cf. Theorem 3.2.6 and Theorem 7.3.1 in [41].

Corollary 5. Let $(X_N : N \in \mathbb{N})$ be a selfadjoint random matrix ensemble. Assume that G_2 admits the integral representation in (5.3). If $f, g : \mathbb{R} \to \mathbb{C}$ are polynomially bounded with $f|_M, g|_M \in C^1([-M, M])$, then

$$\lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left(f(X_N)\right), \mathrm{Tr}\left(g(X_N)\right)\right) = \int_{\mathbb{R}^2} \frac{f(t_2) - f(t_1)}{t_2 - t_1} \frac{g(t_2) - g(t_1)}{t_2 - t_1} \mathrm{d}\nu(t_1, t_2).$$

Note that under the hypotheses of the previous corollary, for $f, g \in C^1([-M, M])$,

$$\rho(f, g) = \int_{[-M,M]} \frac{f(t_2) - f(t_1)}{t_2 - t_1} \frac{g(t_2) - g(t_1)}{t_2 - t_1} \mathrm{d}\nu(t_1, t_2).$$

To the best of the author's knowledge, it is not known whether a bilinear functional on $C^1([-M, M])^2$ satisfying Property b) in Proposition 13 can be written in the above or a similar integral form.

The following theorem shows that Part ii) in Definition 18 leads to the asymptotic Gaussianity of the continuously differential linear statistics.

Theorem 13. Let $(X_N : N \in \mathbb{N})$ be a selfadjoint random matrix ensemble having a second-order limit distribution and satisfying A1 and A2 . If $f : \mathbb{R} \to \mathbb{R}$ is polynomially bounded and $f|_M \in C^1([-M, M])$, then, as $N \to \infty$,

$$\mathrm{Tr}\left(f(X_N)\right) - \mathbb{E}\left(\mathrm{Tr}\left(f(X_N)\right)\right) \Rightarrow \mathcal{N}_{\mathbb{R}}(0, \sigma^2),$$

where $\sigma^2 = \rho(f|_M, f|_M)$.

5.2 Examples and Applications

In this section we gather some examples of random matrix ensembles satisfying A1 and A2. In particular, Example 2 shows that block Gaussian matrices fall under the framework of our main results. Recall that the second-order Cauchy transform of block Gaussian matrices was provided in Chapter 3. In conjunction with Theorem 12, this provides an effective way to compute the covariance of analytic linear statistics of block Gaussian matrices.

Example 1. Let $(X_N : N \in \mathbb{N})$ be the (normalized) Gaussian Unitary Ensemble (GUE), i.e., for each $N \in \mathbb{N}$, X_N is an $N \times N$ selfadjoint random matrix such that $\{X_N(i,j) : 1 \leq i \leq j \leq N\}$ are independent random variables with $X_N(i,i) \sim \mathcal{N}_{\mathbb{R}}(0, N^{-1})$ and $X_N(i,j) \sim \mathcal{N}_{\mathbb{C}}(0, N^{-1})$ $(i \neq j)$. In this case:

0) $(X_N : N \in \mathbb{N})$ has a second-order limit distribution. See [36, Thm 3.1].

1) For all $\epsilon > 0$, there exists $C > 0$ such that

$$\mathrm{P}\left(\|X_N\| > 2(1+\epsilon)\right) \leq 2C \exp\left(-\frac{2\epsilon^2}{C} N\right). \tag{5.4}$$

In particular, A1 is satisfied for every $M > 2$. See [27, eq. 1.4].

2) If $f : \mathbb{R} \to \mathbb{C}$ is differentiable, then

$$\mathrm{Var}\left(\mathrm{Tr}\left(f(X_N)\right)\right) \leq \|f'\|_{\infty}^2. \tag{5.5}$$

This shows that X_N satisfies A2 with $K = 1$. See [41, Prop. 2.1.8].

3) The second-order Cauchy transform G_2 is given by

$$G_2(z,w) = \frac{G'(z)G'(w)}{[G(z)-G(w)]^2} - \frac{1}{(z-w)^2}, \tag{5.6}$$

where $G(z) = \dfrac{z-\sqrt{z^2-4}}{2}$. See [12, eq. 7].

4) The second-order Cauchy transform, as given in (5.6), admits the representation (5.3) with

$$\mathrm{d}\nu(t_1,t_2) = \frac{1}{4\pi^2}\frac{4-t_1t_2}{\sqrt{4-t_1^2}\sqrt{4-t_2^2}}1_{|t_1|\le 2}1_{|t_2|\le 2}\mathrm{d}t_1\mathrm{d}t_2.$$

Note that the relation in (5.6) was established at the level of formal power series. With no rigorous connection to the covariance of resolvents available, it was necessary to prove directly the sought integral representation, as done in [30].

In this case, Theorem 11 recovers Theorem 3.1.1 in [41]. Indeed, after some manipulations, it can be shown that

$$G_2(z,w) = \frac{1}{2(z-w)^2}\left(\frac{zw-4}{\sqrt{z^2-4}\sqrt{w^2-4}} - 1\right).$$

On the other hand, Corollary 5 recovers Theorem 3.2.4 in [41]. Since $(X_N : N \in \mathbb{N})$ has a second-order limit distribution, Theorem 13 implies that the linear statistics of this ensemble are asymptotically Gaussian, cf. Theorem 3.2.6 in [41].

We would like to point out that the Wishart/Laguerre ensemble has a second-order limit distribution and satisfies A1 and A2. See Theorem 3.5 in [36], Theorem 2 in [27], and Proposition 7.2.1 in [41].

Example 2. Let $A_1, \ldots, A_r$ be $d \times d$ selfadjoint matrices. Assume that $X_N^{(1)}, \ldots, X_N^{(r)}$ are independent GUE matrices as in Example 1. The $dN \times dN$ random matrix

$$X_N = \sum_{k=1}^{r} A_k \otimes X_N^{(k)}$$

is called a block Gaussian matrix. In this case:

0) Recall that $\mathrm{Tr}\,(A \otimes B) = \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B)$. Hence

$$\mathrm{Tr}\,(X_N^n) = \sum_{k_1,\ldots,k_n=1}^{r} \mathrm{Tr}\,(A_{k_1} \cdots A_{k_n})\,\mathrm{Tr}\left(X_N^{(k_1)} \cdots X_N^{(k_n)}\right).$$

Since the k_r is multilinear, Theorem 3.1 in [36] readily shows that $(X_N : N \in \mathbb{N})$ has a second-order limit distribution.

1) A routine computation shows that, for every $M > 0$,

$$\mathrm{P}\,(\|X_N\| > M) \leq r\mathrm{P}\left(\left\|X_N^{(1)}\right\| > \frac{M}{r \max_k \|A_k\|}\right).$$

In particular, if we take $M_0 = 4r \max_k \|A_k\|$, then (5.4) implies that

$$\mathrm{P}\,(\|X_N\| > M_0) \leq 2rC \exp\left(-\frac{2}{C}N\right).$$

Thus A1 is satisfied with $M = M_0$.

2) If $f : \mathbb{R} \to \mathbb{C}$ is differentiable, then

$$\mathrm{Var}\,(\mathrm{Tr}\,(f(X_N))) \leq r^2 \left\|\sum_{k=1}^{r} A_k^2\right\|^2 \|f'\|_\infty^2.$$

This shows that block Gaussian matrices satisfy A2. See Proposition 4.7 in [18].

3) The second-order Cauchy transform of a block Gaussian matrix is provided in Theorem 1.

To the best of the author's knowledge, Theorem 13 establishes for the first time a central limit theorem for the linear statistics of block Gaussian matrices. At the moment, it is unknown if the second-order Cauchy transform of these matrices admits the integral representation in (5.3), see the comment after Corollary 5.

Example 3. For each $N \in \mathbb{N}$, let U_N be an $N \times N$ Haar unitary. Assume that $(A_N : N \in \mathbb{N})$ and $(B_N : N \in \mathbb{N})$ are selfadjoint non-random matrix ensembles such that their eigenvalue distributions converge in distribution and $T := \sup_{N\in\mathbb{N}} \max\{\|A_N\|, \|B_N\|\}$ is finite. Let $X_N = A_N + U_N B_N U_N^*$. In this case:

0) $(X_N : N \in \mathbb{N})$ has a second-order limit distribution. See [34, Thm. 1].

1) For all $N \in \mathbb{N}$, $\|X_N\| \leq 2T$. In particular X_N satisfies A1 with $M = 2T$.

2) If $f : \mathbb{R} \to \mathbb{C}$ is differentiable, then

$$\mathrm{Var}\left(\mathrm{Tr}\left(f(X_N)\right)\right) \leq 4T^2 \|f'\|_\infty^2.$$

Thus X_N satisfies A2 with $K = 4T^2$. See [41, pp. 303].

For notational simplicity, let μ_A and μ_B be the limiting eigenvalue distributions of $(A_N : N \in \mathbb{N})$ and $(B_N : N \in \mathbb{N})$, respectively. By Theorem 11,

$$G_2(z, w) = \lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left((z - X_N)^{-1}\right), \mathrm{Tr}\left((w - X_N)^{-1}\right)\right).$$

In notation of [35, Ch. 3], Theorem 10.2.1 and (10.2.30) in [41] imply that

$$G_2(z,w) = \frac{\partial^2}{\partial z \partial w} \log \frac{\omega_A(z) - \omega_A(w)}{z-w} \frac{\omega_B(z) - \omega_B(w)}{F(z) - F(w)},$$

where ω_A and ω_B are the so-called subordination functions and $F(z) = G_{\mu_A \boxplus \mu_B}(z)^{-1}$. This expression for the second-order Cauchy transform is of interest in light of the recent developments in random matrices and free probability based on the subordination functions, e.g., [7, 5, 49, 8].

5.3 Proof of the Main Results

The following notation will be used through the rest of this section. Assume that a selfadjoint random matrix ensemble $(X_N : N \in \mathbb{N})$ is given. For two measurable functions $f, g : \mathbb{R} \to \mathbb{C}$, we define

$$\varphi_N(f) = \frac{1}{N} \mathbb{E} \left(\mathrm{Tr} \left(f(X_N) \right) \right) \quad \text{and} \quad \rho_N(f,g) = \mathrm{Cov} \left(\mathrm{Tr} \left(f(X_N) \right), \mathrm{Tr} \left(g(X_N) \right) \right).$$

Recall that M is the constant from A1. For $f : \mathbb{R} \to \mathbb{C}$, we define $f_M : \mathbb{R} \to \mathbb{C}$ by $f_M(x) = f(x) 1_{|x| \le M}$. The proofs of our main results rely heavily on the following truncation lemma.

Lemma 11. Let $(X_N : N \in \mathbb{N})$ be a selfadjoint random matrix ensemble. Assume that $f, g : \mathbb{R} \to \mathbb{C}$ are measurable functions.

a) If f and g are bounded, then, for all $N \in \mathbb{N}$,

$$|\rho_N(f,g) - \rho_N(f_T, g_T)| \le 4 \|f\|_\infty \|g\|_\infty N^2 \mathrm{P} \left(\|X_N\| > T \right)^{1/4}. \tag{5.7}$$

b) If f and g are polynomially bounded, then there exists $K_{f,g} > 0$ such that, for all $N \in \mathbb{N}$,

$$|\rho_N(f,g) - \rho_N(f_T, g_T)| \leq K_{f,g} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4}. \tag{5.8}$$

Proof. Let $\lambda_1 \leq \cdots \leq \lambda_N$ be the eigenvalues of X_N. Observe that, for any function $h : \mathbb{R} \to \mathbb{C}$,

$$\mathrm{Tr}\left(h(X_N)\right) = \mathrm{Tr}\left(h_T(X_N)\right) + \sum_{i=1}^{N} h(\lambda_i) 1_{|\lambda_i| > T}.$$

In particular, we have that $\rho_N\left(f,g\right) - \rho_N(f_T, g_T) = \mathrm{I} + \mathrm{II} + \mathrm{III}$, where

$$\mathrm{I} = \sum_{j=1}^{N} \mathrm{Cov}\left(\mathrm{Tr}\left(f_T(X_N)\right), g(\lambda_j) 1_{|\lambda_j|>T}\right),$$
$$\mathrm{II} = \sum_{i=1}^{N} \mathrm{Cov}\left(f(\lambda_i) 1_{|\lambda_i|>T}, \mathrm{Tr}\left(g_T(X_N)\right)\right),$$
$$\mathrm{III} = \sum_{i,j=1}^{N} \mathrm{Cov}\left(f(\lambda_i) 1_{|\lambda_i|>T}, g(\lambda_j) 1_{|\lambda_j|>T}\right).$$

By Hölder's inequality, we have that

$$|\mathrm{I}| \leq \mathrm{Var}\left(\mathrm{Tr}\left(f_T(X_N)\right)\right)^{1/2} \sum_{j=1}^{N} \mathrm{Var}\left(g(\lambda_j) 1_{|\lambda_j|>T}\right)^{1/2}.$$

An application of the generalized mean inequality shows that, for any function $h : \mathbb{R} \to \mathbb{C}$,

$$|\mathrm{Tr}\left(h_T(X_N)\right)|^2 \leq N \sum_{i=1}^{N} |h_T(\lambda_i)|^2 \leq N \mathrm{Tr}\left(|h(X_N)|^2\right).$$

Therefore,

$$\operatorname{Var}\left(\operatorname{Tr}\left(f_T(X_N)\right)\right) \leq \mathbb{E}\left(\left|\operatorname{Tr}\left(f_T(X_N)\right)\right|^2\right) \leq N^2\varphi_N(|f|^2),$$

and hence

$$|\mathrm{I}| \leq \varphi_N(|f|^2)^{1/2} N \sum_{j=1}^{N} \operatorname{Var}\left(g(\lambda_j)1_{|\lambda_j|>T}\right)^{1/2}.$$

Hölder's inequality implies that, for all $j \in \{1, \ldots, N\}$,

$$\operatorname{Var}\left(g(\lambda_j)1_{|\lambda_j|>T}\right) \leq \mathbb{E}\left(|g(\lambda_j)|^2 1_{|\lambda_j|>T}\right) \leq \mathbb{E}\left(|g(\lambda_j)|^4\right)^{1/2} \mathrm{P}\left(|\lambda_j| > T\right)^{1/2}.$$

Observe that $\mathrm{P}\left(|\lambda_j| > T\right) \leq \mathrm{P}\left(\|X_N\| > T\right)$. Thus,

$$|\mathrm{I}| \leq \varphi_N(|f|^2)^{1/2} N \mathrm{P}\left(\|X_N\| > T\right)^{1/4} \sum_{j=1}^{N} \mathbb{E}\left(|g(\lambda_j)|^4\right)^{1/4}.$$

Another application of the generalized mean inequality shows that

$$\sum_{j=1}^{N} \mathbb{E}\left(|g(\lambda_j)|^4\right)^{1/4} \leq N\varphi_N(|g|^4)^{1/4}.$$

Therefore,

$$|\mathrm{I}| \leq \varphi_N(|f|^2)^{1/2} \varphi_N(|g|^4)^{1/4} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4}.$$

Mutatis mutandis, it is possible to show that

$$|\mathrm{II}| \leq \varphi_N(|f|^4)^{1/4} \varphi_N(|g|^2)^{1/2} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4},$$

$$|\mathrm{III}| \leq \varphi_N(|f|^4)^{1/4} \varphi_N(|g|^4)^{1/4} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/2}.$$

The last three inequalities imply that

$$|\rho_N(f,g) - \rho_N(f_T, g_T)| \leq K_{f,g,N} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4},$$

where

$$K_{f,g,N} = \left[\varphi_N(|f|^2)^{1/2} + \varphi_N(|f|^4)^{1/4}\right]\left[\varphi_N(|g|^2)^{1/2} + \varphi_N(|g|^4)^{1/4}\right].$$

Now, Part a) is an easy consequence of the fact that, for any function $h : \mathbb{R} \to \mathbb{C}$ and $p > 0$,

$$\varphi_N(|h|^p)^{1/p} = \left(\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left(|h(\lambda_i)|^p\right)\right)^{1/p} \leq \|h\|_\infty.$$

If f and g are polynomially bounded, there exists a polynomial $q : \mathbb{R} \to \mathbb{R}$ such that

$$\max\{|f(x)|^2, |f(x)|^4, |g(x)|^2, |g(x)|^4\} \leq q(x),$$

for all $x \in \mathbb{R}$. In particular, for all $N \in \mathbb{N}$,

$$K_{f,g,N} \leq \left[\varphi_N(q)^{1/2} + \varphi_N(q)^{1/4}\right]^2.$$

By assumption A0, the limit $\lim_{N\to\infty} \varphi_N(q)$ exists. Taking

$$K_{f,g} = \sup\left\{\left[\varphi_N(q)^{1/2} + \varphi_N(q)^{1/4}\right]^2 : N \in \mathbb{N}\right\},$$

Part b) follows. □

By abuse of notation, we denote by $f|_M$ the restriction of $f : \mathbb{R} \to \mathbb{C}$ to the interval

$[-M, M]$. Note that $f|_M$ is a function on $[-M, M]$, while f_M is a function on $\mathbb{R}$. For a function $f : \mathbb{R} \to \mathbb{C}$, we let

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-M, M], \\ f(M) + f'(M)(x - M)e^{-\alpha(x-M)} & x > M, \\ f(-M) + f'(-M)(x + M)e^{\alpha(x+M)} & x < -M, \end{cases}$$

where $\alpha = \|f'\|_M/(e\|f\|_M)$. By construction, $\tilde{f}_M = f_M$. It is not hard to verify that if $f|_M \in C^1([-M, M])$, then

$$\tilde{f} \in C^1(\mathbb{R}), \quad \|\tilde{f}\|_\infty \leq 2\|f\|_M, \quad \text{and} \quad \|\tilde{f}'\|_\infty = \|f'\|_M,$$

where, by abuse of notation, $\|f\|_M = \sup\{|f(x)| : x \in [-M, M]\}$.

Lemma 12. Let $(X_N : N \in \mathbb{N})$ be a selfadjoint random matrix ensemble satisfying A2. Assume that $f, g : \mathbb{R} \to \mathbb{C}$ are measurable functions such that $f|_M, g|_M \in C^1([-M, M])$.

a) If f and g are bounded, then, for all $N \in \mathbb{N}$,

$$|\rho_N(f, g)| \leq 20\|f\|_\infty\|g\|_\infty N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} + K\|f'\|_M\|g'\|_M.$$

b) If f and g are polynomially bounded, then there exists $K_{f,g} > 0$ such that, for all $N \in \mathbb{N}$,

$$|\rho_N(f, g)| \leq K_{f,g} N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} + K\|f'\|_M\|g'\|_M.$$

Proof. Recall that, by construction $\tilde{f}_M = f_M$. In particular,

$$|\rho_N(f,g)| \le |\rho_N(f,g) - \rho(f_M, g_M)| + |\rho_N(\tilde{f}_M, \tilde{g}_M) - \rho_N(\tilde{f}, \tilde{g})| + |\rho_N(\tilde{f}, \tilde{g})|. \quad (5.9)$$

If f and g are bounded, Part a) of Lemma 11 implies that

$$|\rho_N(f,g)| \le 4(\|f\|_\infty \|g\|_\infty + \|\tilde{f}\|_\infty \|\tilde{g}\|_\infty) N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4} + |\rho_N(\tilde{f}, \tilde{g})|.$$

Since $\|\tilde{f}\|_\infty \le 2\|f\|_M \le 2\|f\|_\infty$, we conclude that

$$|\rho_N(f,g)| \le 20\|f\|_\infty \|g\|_\infty N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4} + |\rho_N(\tilde{f}, \tilde{g})|.$$

By A2, we have that $|\rho_N(\tilde{f}, \tilde{g})| \le K\|\tilde{f}'\|_\infty \|\tilde{g}'\|_\infty = K\|f'\|_M \|g'\|_M$. Part a) follows.

If f and g are polynomially bounded, (5.9) and Part b) of Lemma 11 imply that

$$|\rho_N(f,g)| \le (K'_{f,g} + 4\|\tilde{f}\|_\infty \|\tilde{g}\|_\infty) N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4} + |\rho_N(\tilde{f}, \tilde{g})|,$$

for some $K'_{f,g} > 0$. Let $K_{f,g} = K'_{f,g} + 16\|f\|_M \|g\|_M$. In particular,

$$\begin{aligned} |\rho_N(f,g)| &\le K_{f,g} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4} + |\rho_N(\tilde{f}, \tilde{g})| \\ &\le K_{f,g} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4} + K\|f'\|_M \|g'\|_M, \end{aligned}$$

where the last inequality follows from A2 and the fact that $\|\tilde{f}'\|_\infty = \|f'\|_M$. ☐

Proof of Proposition 13. Let $p, q \in \mathbb{C}[x]$. We define

$$\rho(p|_M, q|_M) = \lim_{N \to \infty} \rho_N(p, q).$$

Note that, by A0, the limit in the previous equation exists. By Part b) of Lemma 12, we have that

$$|\rho_N(p,q)| \leq K_{p,q} N^2 \mathrm{P}\left(\|X_N\| > T\right)^{1/4} + K\|p'\|_M \|q'\|_M,$$

for some $K_{p,q} > 0$. Taking limits, A1 implies then

$$|\rho(p|_M, q|_M)| \leq K\|p'\|_M \|q'\|_M.$$

In particular, ρ is continuous with respect to the C^1-norm in $C^1([-M,M])$. A routine argument shows that ρ can be continuously extended (in each argument) to $C^1([-M,M])$. Let $\rho : C^1([-M,M]) \times C^1([-M,M]) \to \mathbb{C}$ denote this extension.

Let $f : \mathbb{R} \to \mathbb{C}$ be a polynomially bounded function with $f|_M \in C^1([-M,M])$ and $q \in \mathbb{C}[x]$. For any polynomial $p \in \mathbb{C}[x]$, we have that $\rho_N(f,q) - \rho(f|_M, q|_M) = \mathrm{I} + \mathrm{II} + \mathrm{III}$, where

$$\mathrm{I} = \rho_N(f,q) - \rho_N(p,q),\ \mathrm{II} = \rho_N(p,q) - \rho(p|_M, q|_M),\ \mathrm{III} = \rho(p|_M, q|_M) - \rho(f|_M, q|_M).$$

By Part b) of Lemma 12, we have that

$$|\mathrm{I}| \leq K_{f-p,q} N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} + K\|(f-p)'\|_M \|q'\|_M,$$

for some $K_{f-p,q} > 0$. Let $\epsilon > 0$. By the density of the polynomials in $C^1([-M,M])$ with respect to the C^1-norm and the continuity of ρ with respect to the same norm, there exists $p_0 \in \mathbb{C}[x]$ such that

$$\|(f-p_0)'\|_M \leq \frac{\epsilon}{6K\|q'\|_M} \quad \text{and } |\text{III}| \leq \frac{\epsilon}{3}.$$

By construction, $\rho_N(p_0, q) \to \rho((p_0)|_M, q|_M)$ as $N \to \infty$. Combined with A1, this implies that there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$|\text{II}| \leq \frac{\epsilon}{3} \quad \text{and } K_{f-p_0,q} N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} \leq \frac{\epsilon}{6}.$$

Therefore, for all $N > N_0$, $|\rho_N(f,q) - \rho(f|_M, q|_M)| \leq \epsilon$, i.e.,

$$\lim_{N\to\infty} \rho_N(f,q) = \rho(f|_M, q|_M).$$

The previous equation and a similar argument show that

$$\lim_{N\to\infty} \rho_N(f,g) = \rho(f|_M, g|_M)$$

for all polynomially bounded functions $f, g : \mathbb{R} \to \mathbb{C}$ with $f|_M, g|_M \in C^1([-M, M])$. This proves Part b). Part a) is an easy consequence of Part b) of Lemma 12. $\square$

For $z \in \mathbb{C}$, we let $r_z : \mathbb{C} \setminus \{z\} \to \mathbb{C}$ be given by $r_z(x) = \dfrac{1}{z-x}$. We define $G_{2,M}^{(N)} : (\mathbb{C} \setminus [-M, M])^2 \to \mathbb{C}$ by

$$G_{2,M}^{(N)}(z,w) = \rho_N(r_{z,M}, r_{w,M}), \tag{5.10}$$

where $r_{z,M} = (r_z)_M$. It is not hard to verify that $G_{2,M}^{(N)}$ is analytic. Moreover, it satisfies the following boundedness property.

Lemma 13. If $(X_N : N \in \mathbb{N})$ is a selfadjoint random matrix ensemble satisfying A1 and A2, then for every $z_0, w_0 \in \mathbb{C} \setminus [-M, M]$ there exists $\delta > 0$ such that

$$\sup\left\{\left|G_{2,M}^{(N)}(z,w)\right| : |z - z_0| < \delta, |w - w_0| < \delta, N \in \mathbb{N}\right\} < \infty.$$

Proof. Note that, for all $z \in \mathbb{C} \setminus [-M, M]$, the function $r_{z,M} : \mathbb{R} \to \mathbb{C}$ satisfies

$$\|r_{z,M}\|_\infty = d(z)^{-1} \quad \text{and} \quad \|r'_{z,M}\|_M = d(z)^{-2},$$

where $d(z) := \inf\{|z - x| : x \in [-M, M]\}$. By Part a) of Lemma 12, for all $z, w \in \mathbb{C} \setminus [-M, M]$,

$$\begin{aligned}|G_{2,M}^{(N)}(z,w)| &\leq 20\|r_{z,M}\|_\infty \|r_{w,M}\|_\infty N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} + K\|r'_{z,M}\|_M \|r'_{w,M}\|_M \\ &= 20 d(z)^{-1} d(w)^{-1} N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} + K d(z)^{-2} d(w)^{-2}.\end{aligned}$$

Let $\delta = \frac{1}{2}\min(d(z_0), d(w_0))$. Note that for all $z, w \in \mathbb{C}$ such that $|z - z_0| < \delta$ and $|w - w_0| < \delta$, we have that $d(z) > \delta$ and $d(w) > \delta$. Assumption A1 implies that $\{N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} : N \in \mathbb{N}\}$ is bounded. Therefore,

$$\sup\left\{\left|G_{2,M}^{(N)}(z,w)\right| : |z - z_0| < \delta, |w - w_0| < \delta, N \in \mathbb{N}\right\} < \infty,$$

as required. □

Lemma 14. If $(X_N : N \in \mathbb{N})$ is a selfadjoint random matrix ensemble satisfying A1 and A2, then the family $\left\{G_{2,M}^{(N)} : N \in \mathbb{N}\right\}$ converges uniformly in compact subsets of

$(\mathbb{C} \setminus \mathbb{R})^2$ and, for all $|z|, |w| > M$,

$$\lim_{N\to\infty} G_{2,M}^{(N)}(z,w) = G_2(z,w).$$

Proof. By definition,

$$G_{2,M}^{(N)}(z,w) = \operatorname{Cov}\left(\sum_{i=1}^{N} \frac{1_{|\lambda_i|\le M}}{z-\lambda_i}, \sum_{j=1}^{N} \frac{1_{|\lambda_j|\le M}}{w-\lambda_j}\right).$$

If $|z|, |w| > M$, then we have that

$$(z-\lambda_i)^{-1} = \sum_{m\ge 0} \frac{\lambda_i^m}{z^{m+1}} \quad \text{and} \quad (w-\lambda_j)^{-1} = \sum_{n\ge 0} \frac{\lambda_j^n}{w^{n+1}}.$$

In particular, for such z and w, we have that

$$G_{2,M}^{(N)}(z,w) = \operatorname{Cov}\left(\sum_{i=1}^{N}\sum_{m\ge 0} \frac{\lambda_i^m 1_{|\lambda_i|\le M}}{z^{m+1}}, \sum_{j=1}^{N}\sum_{n\ge 0} \frac{\lambda_j^n 1_{|\lambda_j|\le M}}{w^{n+1}}\right).$$

For each $k \in \mathbb{N}$, let $\pi^{(k)}$ denote the function given by $\pi^{(k)}(x) = x^k$. With this notation, we can rewrite the previous equation as

$$G_{2,M}^{(N)}(z,w) = \operatorname{Cov}\left(\sum_{m\ge 0} \frac{\operatorname{Tr}\left(\pi_M^{(m)}(X_N)\right)}{z^{m+1}}, \sum_{n\ge 0} \frac{\operatorname{Tr}\left(\pi_M^{(n)}(X_N)\right)}{w^{n+1}}\right).$$

Since $|\operatorname{Tr}\left(\pi_M^{(k)}(X_N)\right)| \le NM^k$, a routine application of the Tonelli-Fubini theorem implies that

$$G_{2,M}^{(N)}(z,w) = \sum_{m,n\ge 0} \frac{\rho_N(\pi_M^{(m)}, \pi_M^{(n)})}{z^{m+1}w^{n+1}}.$$

Note that $\pi_M^{(k)}$ satisfies that

$$\|\pi_M^{(k)}\|_\infty = M^k \quad \text{and} \quad \|(\pi_M^{(k)})'\|_M = kM^{k-1}.$$

By Part a) of Lemma 12, we obtain

$$|\rho_N(\pi_M^{(m)}, \pi_M^{(n)})| \leq 20M^{m+n}N^2\mathrm{P}\left(\|X_N\| > M\right)^{1/4} + mnKM^{m+n-2}.$$

By A1, the set $\left\{N^2\mathrm{P}\left(\|X_N\| > M\right)^{1/4} : N \in \mathbb{N}\right\}$ is bounded. Hence,

$$|\rho_N(\pi_M^{(m)}, \pi_M^{(n)})| \leq mnBM^{m+n},$$

for some constant $B > 0$. The previous inequality and the dominated convergence theorem imply that

$$\lim_{N\to\infty} G_{2,M}^{(N)}(z,w) = \sum_{m,n\geq 0} \lim_{N\to\infty} \frac{\rho_N(\pi_M^{(m)}, \pi_M^{(n)})}{z^{m+1}w^{n+1}},$$

for all $|z|, |w| > M$. Part b) of Proposition 13 implies that

$$\lim_{N\to\infty} \rho_N(\pi_M^{(m)}, \pi_M^{(n)}) = \rho(\pi^{(m)}|_M, \pi^{(n)}|_M) = \alpha_{m,n}.$$

In other words, for all $|z|, |w| > M$,

$$\lim_{N\to\infty} G_{2,M}^{(N)}(z,w) = G_2(z,w). \tag{5.11}$$

By the previous lemma, the family $\left\{G_{2,M}^{(N)} : N \in \mathbb{N}\right\}$ is locally bounded. Therefore, Montel's theorem [47, pp. 33] and (5.11) imply that $\left\{G_{2,M}^{(N)}(z,w) : N \in \mathbb{N}\right\}$ converges towards an analytic function on $(\mathbb{C}\setminus[-M,M])^2$, i.e., it converges uniformly on compact sets. $\square$

For notational simplicity, for each $z, w \in \mathbb{C} \setminus \mathbb{R}$, we let

$$G_2^{(N)}(z,w) = \mathrm{Cov}\left(\mathrm{Tr}\left((z - X_N)^{-1}\right), \mathrm{Tr}\left((w - X_N)^{-1}\right)\right).$$

Note that $G_2^{(N)}(z,w) = \rho_N(r_z, r_w)$. Now we are in position to prove our main results.

Proof of Theorem 11. By the previous lemma, $\left\{G_{2,M}^{(N)} : N \in \mathbb{N}\right\}$ converges uniformly in compact subsets of $(\mathbb{C} \setminus [-M,M])^2$ to an analytic function. Since

$$G_2(z,w) = \lim_{N\to\infty} G_{2,M}^{(N)}(z,w)$$

for all $|z|, |w| > M$, we agree to denote by G_2 such an extension. By Part a) of Lemma 11, we have that

$$\left|G_2^{(N)}(z,w) - G_{2,M}^{(N)}(z,w)\right| \leq 4\|r_z\|_\infty \|r_w\|_\infty N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4}.$$

Since $\|r_z\|_\infty \leq |\Im z|^{-1}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, A1 implies that

$$\lim_{N\to\infty} G_2^{(N)}(z,w) = \lim_{N\to\infty} G_{2,M}^{(N)}(z,w) = G_2(z,w),$$

as required. $\square$

Proof of Theorem 12. Whenever $|\lambda_i| \leq M$, Cauchy's integral formula implies that

$$f(\lambda_i) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{z-\lambda_i}\mathrm{d}z.$$

In particular, we have that

$$\rho_N(f_M, g_M) = \sum_{i,j=1}^{N} \mathrm{Cov}\left(\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)1_{|\lambda_i|\leq M}}{z-\lambda_i}\mathrm{d}z, \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{g(w)1_{|\lambda_j|\leq M}}{w-\lambda_j}\mathrm{d}w\right).$$

A routine application of the Tonelli-Fubini theorem shows then that

$$\begin{aligned}\rho_N(f_M, g_M) &= \sum_{i,j=1}^{N} \frac{1}{(2\pi i)^2}\int_{\mathcal{C}}\int_{\mathcal{C}} f(z)g(w)\mathrm{Cov}\left(\frac{1_{|\lambda_i|\leq M}}{z-\lambda_i}, \frac{1_{|\lambda_j|\leq M}}{w-\lambda_j}\right)\mathrm{d}z\mathrm{d}w \\ &= \frac{1}{(2\pi i)^2}\int_{\mathcal{C}}\int_{\mathcal{C}} f(z)g(w)G_{2,M}^{(N)}(z,w)\mathrm{d}z\mathrm{d}w.\end{aligned}$$

By Lemma 14, $\left\{G_{2,M}^{(N)} : N \in \mathbb{N}\right\}$ converges uniformly to (the extension of) G_2 in the compact set $\mathcal{C} \times \mathcal{C}$. Therefore, by the dominated convergence theorem,

$$\lim_{N\to\infty} \rho_N(f_M, g_M) = \frac{1}{(2\pi i)^2}\int_{\mathcal{C}}\int_{\mathcal{C}} f(z)g(w)G_2(z,w)\mathrm{d}z\mathrm{d}w.$$

By Part b) of Proposition 13, we conclude that

$$\lim_{N\to\infty} \rho_N(f,g) = \rho(f|_M, g|_M) = \lim_{N\to\infty} \rho_N(f_M, g_M).$$

Equation (5.2) follows. □

Proof of Corollary 5. Recall that the integral representation in (5.3) assumes that

$$\mathrm{Supp}\,(\nu) \subset [-M, M].$$

Let $I : C^1([-M, M])^2 \to \mathbb{C}$ be the bilinear functional given by

$$I(f, g) = \int_{[-M,M]^2} \frac{f(t_1) - f(t_2)}{t_1 - t_2} \frac{g(t_1) - g(t_2)}{t_1 - t_2} \mathrm{d}\nu(t_1, t_2).$$

Note that $|I(f, g)| \leq |\nu|(\mathbb{R}^2)\|f'\|_M \|g'\|_M$, i.e., I is a continuous in each argument with respect to the C^1-norm in $C^1([-M, M])$.

Let $p, q \in \mathbb{C}[x]$. By Proposition 13 and Theorem 12,

$$\rho(p|_M, q|_M) = \lim_{N\to\infty} \rho_N(p, q) = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}} \int_{\mathcal{C}} p(z)q(w)G_2(z, w)\mathrm{d}z\mathrm{d}w.$$

Since G_2 admits the integral representation in (5.3), Theorem 6 implies that

$$\rho(p|_M, q|_M) = \int_{[-M,M]^2} \frac{p(t_1) - p(t_2)}{t_1 - t_2} \frac{q(t_1) - q(t_2)}{t_1 - t_2} \mathrm{d}\nu(t_1, t_2) = I(p|_M, q|_M).$$

Since both ρ and I are continuous in each argument with respect to the C^1-norm and they agree on polynomials, a standard argument shows that $\rho = I$. In particular,

$$\begin{aligned}\lim_{N\to\infty} \mathrm{Cov}\left(\mathrm{Tr}\left(f(X_N)\right), \mathrm{Tr}\left(g(X_N)\right)\right) &= \rho(f|_M, g|_M) \\ &= \int_{\mathbb{R}^2} \frac{f(t_1) - f(t_2)}{t_1 - t_2} \frac{g(t_1) - g(t_2)}{t_1 - t_2} \mathrm{d}\nu(t_1, t_2),\end{aligned}$$

for all $f, g : \mathbb{R} \to \mathbb{C}$ polynomially bounded with $f, g \in C^1([-M, M])$. □

The following lemma is an easy consequence of Runge's theorem and the Schwarz

Reflection Principle. For a set $S \subset \mathbb{C}$, we let $S^* = \{z \in \mathbb{C} : \overline{z} \in S\}$.

Lemma 15. Let $\Omega \subset \mathbb{C}$ be a domain. Assume that $\mathcal{K} = \mathcal{K}^*$ is a compact subset of Ω whose complement is connected. If $f : \Omega \to \mathbb{C}$ is an analytic function such that $f(\mathbb{R}) \subset \mathbb{R}$, then there exist polynomials $(p_k : k \in \mathbb{N}) \subset \mathbb{R}[z]$ such that

$$\lim_{k\to\infty} \|p_k - f\|_{\mathcal{K}} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|(p_k - f)'\|_{\mathcal{K}} = 0,$$

where $\|g\|_{\mathcal{K}} = \sup\{|g(z)| : z \in \mathcal{K}\}$.

Proof of Theorem 13. Let $\mathcal{K} \subset \Omega$ be a compact set as in Lemma 15 that contains $\mathcal{C}$ and $[-M, M]$. By Lemma 15, there exist real polynomials $(p_k : k \in \mathbb{N})$ such that

$$\lim_{k\to\infty} \|p_k - f\|_{\mathcal{K}} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|(p_k - f)'\|_{\mathcal{K}} = 0.$$

For notation simplicity, for all $k \in \mathbb{N}$ and $N \in \mathbb{N}$, we let

$$Z_N^{(k)} = \operatorname{Tr}\left(p_k(X_N)\right) - \mathbb{E}\left(\operatorname{Tr}\left(p_k(X_N)\right)\right).$$

Clearly $k_1\left(Z_N^{(k)}\right) = 0$. From ii) in Definition 18 (p. 90), it is immediate to see that, for all $r \geq 3$,

$$\lim_{N\to\infty} k_r\left(Z_N^{(k)}, \ldots, Z_N^{(k)}\right) = 0.$$

Since the Gaussian distribution on $\mathbb{R}$ is characterized by its moments, and hence by its cumulants, we conclude from Proposition 13 that $Z_N^{(k)} \Rightarrow Z^{(k)} \sim \mathcal{N}_{\mathbb{R}}(0, \sigma_k^2)$, where

$\sigma_k^2 = \rho(p_k|_M, p_k|_M)$. By Property a) in the same proposition,

$$|\sigma_k^2 - \sigma^2| \leq |\rho(p_k|_M, p_k|_M) - \rho(p_k|_M, f|_M)| + |\rho(p_k|_M, f|_M) - \rho(f|_M, f|_M)|$$
$$\leq K(\|p_k'\|_{\mathcal{K}} + \|f'\|_{\mathcal{K}})\|(p_k - f)'\|_{\mathcal{K}}.$$

In particular, $\sigma_k^2 \to \sigma^2$ and hence $Z^{(k)} \Rightarrow Z \sim \mathcal{N}_{\mathbb{R}}(0, \sigma^2)$.

For all $N \in \mathbb{N}$, we let $Z_N = \mathrm{Tr}\,(f(X_N)) - \mathbb{E}\,(\mathrm{Tr}\,(f(X_N)))$. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function. Note that for all $N \in \mathbb{N}$

$$\mathbb{E}\,(g(Z_N)) - \mathbb{E}\,(g(Z)) = \alpha_{k,N} + \beta_{k,N} + \gamma_k$$

for every $k \in \mathbb{N}$, where $\alpha_{k,N} = \mathbb{E}\left(g(Z_N) - g\left(Z_N^{(k)}\right)\right)$, $\beta_{k,N} = \mathbb{E}\left(g\left(Z_N^{(k)}\right) - g\left(Z^{(k)}\right)\right)$, and $\gamma_k = \mathbb{E}\left(g\left(Z^{(k)}\right) - g(Z)\right)$. Let L_g be a Lipschitz constant for g. A routine computation shows that

$$|\alpha_{k,N}| \leq L_g \mathbb{E}\left(\left|Z_N - Z_N^{(k)}\right|\right) \leq L_g \mathrm{Var}\left(Z_N - Z_N^{(k)}\right)^{1/2}.$$

Observe that

$$\mathrm{Var}\left(Z_N - Z_N^{(k)}\right) = \mathrm{Var}\,(\mathrm{Tr}\,((p_k - f)(X_N))) = \rho_N(p_k - f, p_k - f).$$

By Part b) of Lemma 12, there exists $C_k > 0$ such that, for all $N \in \mathbb{N}$,

$$|\rho_N(p_k - f, p_k - f)| \leq C_k N^2 \mathrm{P}\,(\|X_N\| > M)^{1/4} + K\|(p_k - f)'\|_{\mathcal{K}}^2.$$

In particular, we have that

$$|\alpha_{k,N}| \leq L_G \left(C_k N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} + K\|(p_k - f)'\|_{\mathcal{K}}^2 \right)^{1/2}.$$

Let $\epsilon > 0$. Since $Z^{(k)} \Rightarrow Z$, we have that $\mathbb{E}\left(g\left(Z^{(k)}\right)\right) \to \mathbb{E}\left(g(Z)\right)$ as $k \to \infty$. Let $k_0 \in \mathbb{N}$ be such that

$$|\gamma_{k_0}| = \left|\mathbb{E}\left(g\left(Z^{(k_0)}\right)\right) - \mathbb{E}\left(g(Z)\right)\right| \leq \frac{\epsilon}{3} \quad \text{and} \quad \|(p_k - f)'\|_{\mathcal{K}}^2 \leq \frac{\epsilon^2}{18KL_g^2}.$$

Since $N^8 \mathrm{P}\left(\|X_N\| > M\right) \to 0$ and $Z_N^{(k_0)} \Rightarrow Z^{(k_0)}$ as $N \to \infty$, there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$

$$C_{k_0} N^2 \mathrm{P}\left(\|X_N\| > M\right)^{1/4} \leq \frac{\epsilon^2}{18L_g^2} \text{ and } |\beta_{k_0,N}| = \left|\mathbb{E}\left(g\left(Z_N^{(k_0)}\right)\right) - \mathbb{E}\left(g\left(Z^{(k_0)}\right)\right)\right| \leq \frac{\epsilon}{3}.$$

Therefore, $|\mathbb{E}\left(g(Z_N)\right) - \mathbb{E}\left(g(Z)\right)| \leq \epsilon$ for all $N > N_0$, i.e.,

$$\lim_{N\to\infty} \mathbb{E}\left(g(Z_N)\right) = \mathbb{E}\left(g(Z)\right).$$

The Portmanteau lemma implies then that $Z_N \Rightarrow Z$. In other words,

$$\mathrm{Tr}\left(f(X_N)\right) - \mathbb{E}\left(\mathrm{Tr}\left(f(X_N)\right)\right) \Rightarrow \mathcal{N}_{\mathbb{R}}(0, \sigma^2),$$

as we wanted to prove. ∎

Chapter 6

Conclusions

In this thesis we proved that the continuously differentiable linear statistics of a selfadjoint random matrix ensemble exhibit a Central Limit Theorem as long as the matrix norm of the ensemble satisfies a weak form of a large deviation principle, the linear statistics of the ensemble obey a Poincaré-type inequality, and the ensemble itself has a second-order limit distribution. In addition, we showed that the covariance of the traces of resolvents converges to the second-order Cauchy transform and that the covariance of the traces of analytic linear statistics converges to a contour integral depending on the second-order Cauchy transform. The latter fact emphasizes the importance of this transform in random matrix theory terms.

Motivated by the above discussion, we computed the second-order Cauchy transform associated to selfadjoint block Gaussian matrices. In the process, we introduced a second-order conditional expectation with values over the complex matrices and computed the matricial generating function associated to a new type of pairings that we called double-line.

Also, we pursued a particular integral representation for the second-order Cauchy transform which is common in the literature. In this direction, we proved that the positivity of a cluster function implies the veracity of the desired integral representation, which in turn provides a similar expression for the limit of the covariance of the traces of continuously differentiable linear statistics.

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